Regiomontanus and the First Optimization Problem Since Antiquity?

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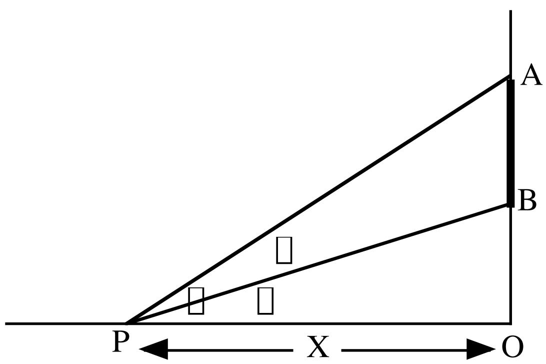
The First Optimization Problem since Antiquity?

In 1471, Johann Muller, alias Regiomontanus, posed the question: At what horizontal distance from an elevated rod would a person have to stand such that the appearance of the rod should be a maximum?

Presenting this problem in terms of data analysis can easily be done by measuring the inscribed angles formed between the “ground” and the rod and constructing a table of values with x the horizontal distance from the bottom of the suspended rod, and y the measure of the angle. Students are asked to write a function that will generate the table using the angles □ and □ given in the diagram below. This exercise is introduced after the inverse tangent function is taught but before the tangent of the difference of two angles.



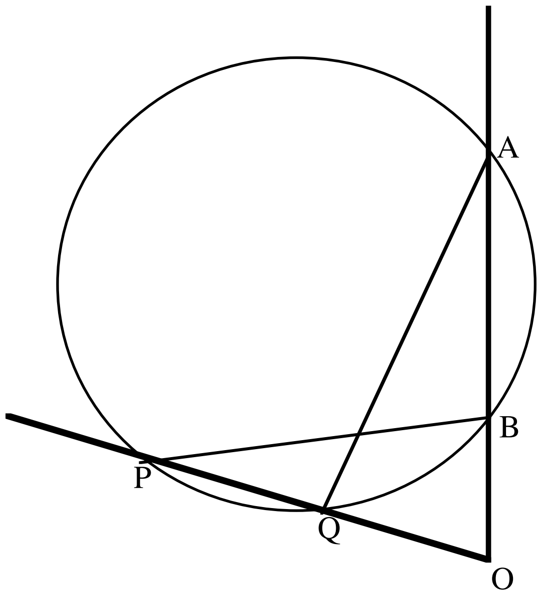
Working equation: □ = tan-1AO/X - tan-1BO/X, where tan□ = AO/X and tan □ = BO/X. AO and BO are given quantities. Define AO = A, and BO = B

Calculator equation: y = Tan-1A/X - Tan-1B/X, is used to fit the data.

Extending the problem. Eli Moar, in his book, <u>Trigonometric Delights</u>, states he was unable to determine whether a solution by Regiomontanus exists. Moar does however offer the following argument based on elementary methods that would have been available to Regiomontanus.

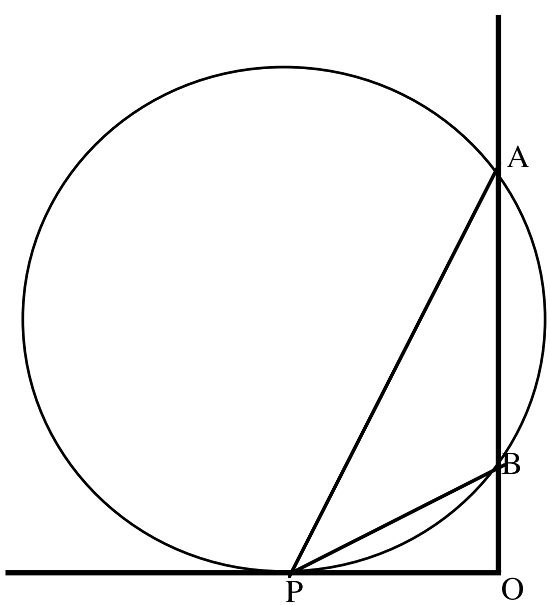
cot □ = (cot □ cot □ +1) / (cot □ - cot □). Replace cot □ with X/A, and cot □ = X/B. After a bit of algebra the cot □ can be expressed as X/(A - B) + AB / [(A - B)X]. The advantage of expressing cot □ as the sum of two terms is to take advantage of a theorem from algebra that states that the arithmetic mean of two positive numbers is never smaller than their geometric mean. That is, (c + d)/2 is greater than or equal to √(cd). This can be seen if we note $(\sqrt{c} - \sqrt{d})^2$ is greater than or equal to zero, in other words the square of a real number can never be negative. If we let X/(A- B) = c and AB / [(A - B)X] = d, then cot □ = c + d. But recall, the cotangent function decreases over the interval (0º, 90 º]. Therefore, a maximum angle--which is what we’re looking for--will occur for a minimum value of cot □. (c + d)/2 has its minimum when it equals √(cd) which can only occur when c = d, in which case, X/(A - B) = AB / [(A - B)X], giving X = √(AB). In other words, the optimum distance for viewing is the geometric mean of A and B.

A final word: There is a theorem in geometry regarding a circle being cut by two secants: Note the next diagram.



Consider the two secants: OA and OP, and the two triangles: QOA and POB. Both have the angle POA in common, and the angles QPB and BAQ both have the arc QB in common so these angles are equal also. Hence, triangles QOA and POB are similar. Therefore: OP/OB = OA/ OQ

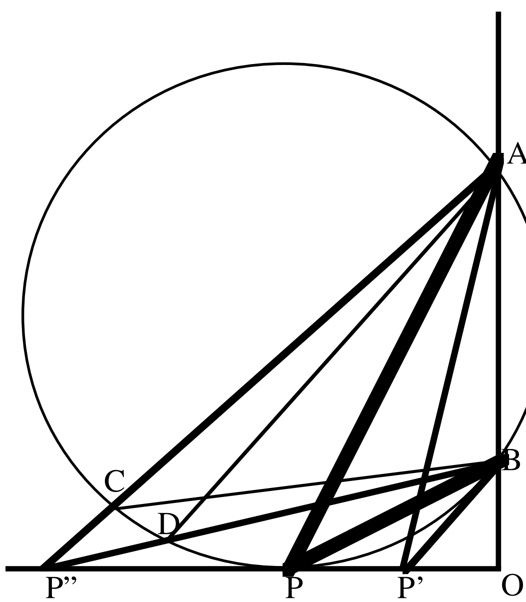
Consider what happens as the circle increases and the secant OP becomes perpendicular to the secant OA, the point P becoming tangent to the circle. The diagram becomes:



A comparison of the two diagrams gives OB/OP = OP/OA or $OP^2 = OA\ OB$,
that is $x^2 = AB$.

Lastly, the entire problem could have been done purely geometrically in a 10th grade geometry class. Consider the reasoning below.

Recall from geometry that if point P is moved about the circle the angle APB remains constant (the arc AB is fixed). Therefore, for any point other than the tangent point P, the angle APB will be smaller if P remains on the line OP since to the right or left of P the segments AP', BP' and AP", BP" are farther to the tangent line OP than to the circle. Thus the maximum angle condition, for a viewer on the ground, is achieved when P is tangent to the circle. See Diagram below.



Angle AP"B < Angle APB and Angle AP'B < Angle APB, since angles ACB, ADB and APB are equal to each other and greater than angles AP"B and AP'B. Further constructions with the circle can be done near the triangle AP'B showing the same results.

Recommended reading: Trigonometric Delights by Eli Maor, Princeton University Press.