When Linearity Isn't Enough

The world curves, space warps, populations explode, and death abruptly halts a thousand unseen miracles—nothing in the physical world is linear. It is only humanity's lack of perception that has formed linear relationships out of the small piece of reality it is privy to.

Anyone who has ever traveled across the Texas Panhandle can appreciate why, for most of human history, the world was thought to be flat. The illusion is difficult to overcome when faced with a small view of a large landscape. The surface area of the Earth is roughly 2×10^8 square miles (two hundred million square miles). From a height of six feet above the Earth's surface we can see only an area of about thirty square miles. This perspective easily transforms the round Earth into a flat plane. Though such a conclusion is not unreasonable, careful observation suggests otherwise. Only the peak of a mountain is visible from a great distance, the rest is consumed by the curvature of the Earth. Likewise, ships far out at sea have hidden hulls, while full sails still remain in sight.

A circle is certainly far removed from a straight line and hence, definitely not linear. But imagine taking a very small segment out of a very large circle. If the small segment is seen in isolation, it can be mistakenly assumed a straight line. Everything depends upon how much of reality is accessible to measurement. The smaller the domain, the more accurately nonlinear relationships can be approximated with linear ones.

The word "domain" actually has a very precise mathematical definition. The domain represents all possible values for the independent variable x. Similarly, the "range" stands for all values of the dependent variable y. All functions have a domain and range whether they are linear or not.

In Problem 2 in Chapter Seven, the dependent variable (ppm of atmospheric carbon dioxide) is plotted against the independent variable of time. The domain covered 9 years [1959,1968] and the range 6.3 ppm [316.2,322.5]. The graph of this function is reproduced in Figure 8.1 with the dashed line representing linear extrapolations into the past and future where no data was originally given. On the same graph, more of the function is plotted over the enlarged domain [1959,1991].

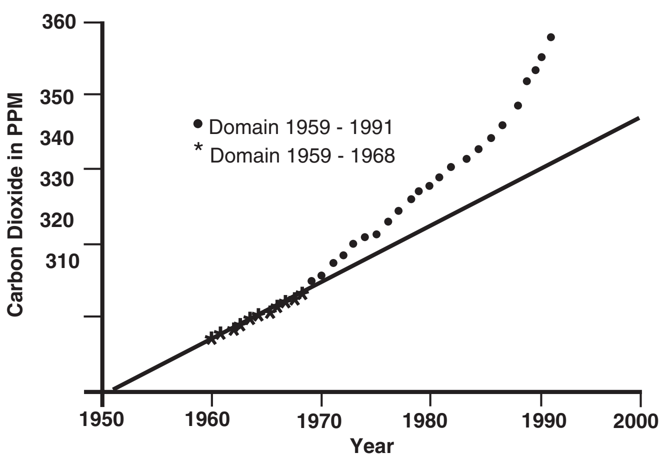

Figure 8.1

Atmospheric carbon dioxide concentration appears linear if viewed over the restricted domain 1959-1968. The dashed line represents the linear extrapolation of the restricted domain through the year 2000. Note how the actual data over the enlarged domain diverges from the linear model.

Source: Adapted from *Vital Signs 1992* Worldwatch Institute.

Clearly, the amount of atmospheric carbon dioxide has not been growing linearly with time. Therefore, any linear extrapolations of this function over more than a limited domain, would lead to ever increasing errors. Care must always be taken when using mathematical models to predict future trends. A mathematical model is not a complete description of reality; it represents a relationship between the variables that we have measured and our common sense understanding of the problem. To ask for more exceeds the bounds of the scientific method.

All of us have a tendency to reduce complex situations into simple models. These models may be adequate for small scale interpretations, but usually fail when extended. Perception without knowledge leads to dubious conclusions at best. As we have seen, circles become lines and spheres turn in to a planes. Additionally, a small life span, compared to epochs needed for biological and geological change, makes for an apparently static world. Our perception is constrained to the smallest tick of the cosmic clock. Connected events that span decades and centuries easily become isolated events to us. Small scale linear thinking sets in and the "big picture" is overlooked. If such were not the case, the environment would be healthy; a clear understanding between diet, health, and the environment would guide peoples' food choices; affordable health care would exist; and those polices that fostered our growing national debt would never have been implemented.

There is an interesting experiment called the Boiled Frog Syndrome.[1] It turns out that when a frog is placed in water and the temperature slowly increased, he is unable to detect any danger. Unless he is removed, he will remain in the water and eventually die. The same mind-set exists with cigarette smokers, drug users, and corporate polluters. The more narrow and linear a series of events appears to us, the less apt we are to take action. If negative change occurs too slowly, we do nothing but instinctively accommodate ourselves to an ever worsening situation.

Few times in human history has an understanding of nonlinear principles been so important. During the twentieth century we have been busy collecting data on the effects of industrialization on the ecosphere and on human health. We have learned that ecological changes do not follow linear sequential processes. Similarly, the biological changes which occur within each of us are also nonlinear. Regrettably, we have a difficult time perceiving, and hence understanding, these changes. A father dies at age sixty from a heart attack. Family, friends, and business associates are shocked because, "He was always so healthy and strong as an ox." But in addition to the stress at work, he ate lots of animal foods and smoked a few cigarettes each night. And then, after a particularly stressful morning at work, a hurried burger and fries for lunch, and a few cigarettes before retiring for the evening, the "straw that broke the camel's back" hit hard. The cumulative nonlinear effects of a lifetime of stress and bad habits precipitated the fatal heart attack. Processes in nature are similar. Nonlinearity and abrupt change are how nature defines itself, but such characteristics are never understood if the domain of observation is too narrow. How can scientists convince politicians and the general public of the seriousness of a situation, if deterioration initially appears modest? Without an appreciation of the true nature of change, and the mathematics that describe the rate of such change, linear thinking may result in ignoring timely precautions. In an age where massive oil spills, deforestations, and rapidly rising human population and resource consumption are common, understanding nonlinearity is essential.

The Parabola as an Example of Nonlinearity

Both linear and nonlinear functions can change quickly or slowly. The major difference is that linear functions have a constant rate of growth or decay, whereas nonlinear functions do not. The rate of change for a nonlinear function is itself always changing; additionally, this change may or may not be linear.

The salary example, $y = \$5.00t$, is a linear function because the

rate of pay is constant at $5.00 per hour. Regardless of the number of hours worked, the pay rate remains $5.00 each hour. Figure 8.2a shows the rate of pay and Figure 8.2b shows the earnings.

The rate of pay graph is a horizontal line which means it is a constant function. One could think of this graph as analogously representing a constant speed of five miles per hour, and the earnings graph as the increasing distance traveled over an eight hour period at this constant rate. If we step on the accelerator the situation becomes nonlinear. As we accelerate, the distance traveled during each successive equal time interval grows larger, because the speed at which we travel continually increases. (Try constructing a table of values for figures 8.2a and 8.2b showing time, money earned and pay rate.)

Similarly, the linear earnings example of Figure 8.2b can be transformed into a nonlinear relationship by uniformly increasing the rate of pay by a fixed amount (say $5.00 per hour) every hour. It is important to understand what is meant by a "uniformly increasing rate." The pay rate does not abruptly change from $5.00/h to $10.00/h at the beginning of the second hour of work. Rather, the rate is smoothly and continuously changing from moment to moment, until at the beginning of the second hour it reaches a rate of $10.00/h. So the salary uniformly increases from an initial amount of $5.00/h, to $10.00/h, to $15.00/h, and so on, for each successive hour. Table 8.1 gives the earnings (middle column) and pay rate (last column) for an eight hour day.

Table 8.1

Earnings and pay rate over an eight hour period for a pay rate that begins at $5.00 per hour and is increased uniformly at $5.00 per hour every hour.

Time(hours)	Money Earned(dollars)	Pay Rate(dollars/h)
0	0.00	5
1	7.50	10
2	20.00	15
3	37.50	20

Time(hours)	**Money Earned(dollars)**	**Pay Rate(dollars/h)**
4	60.00	25
5	87.50	30
6	120.00	35
7	157.50	40
8	200.00	45

The equation describing the money earned is: $y = 2.5t^2 + 5t$, where y represents the earnings and t the time worked. For example, when $t = 6$ hours we substitute 6 for t which gives:

$2.5(6)^2 + 5(6) = 120$ dollars (see Table 8.1 above).

The equation for the rate of pay, however, is expressed by the linear equation: $y = 5t + 5$, where y represents the pay rate and t the time worked. Notice at $t = 0$ we get the initial condition that the pay rate is \$5.00/hour: $y = 5(0) + 5 =$ \$5.00/hour.

It is a legitimate question to ask how the earnings equation is derived. Regrettably, the earnings equation is nonlinear and the derivation for such an equation requires considerable mathematical background. Readers who are up to the challenge are encouraged to read Appendix C. Fortunately, it is not necessary to understand the derivation for this equation in order to see the logic of its results. Since the pay rate changes uniformly, the average salary during the first hour of work is \$7.50/h (the average of \$5.00/h and \$10.00/h). Similarly, the average pay rate for the second time interval is \$12.50/h (the average of \$10.00/h and \$15.00/h). Since \$12.50 is earned during this time interval, the total money earned from the start of work through the second hour is \$7.50 + \$12.50 = \$20.00. The same line of reasoning will produce all the other values in the table.

Notice that the earnings equation has the squared term $2.5t^2$ in it. Such equations are called quadratic equations. The exact definition is: **an equation having a squared term as its largest power is called a second degree equation or quadratic equation. The graph of a quadratic equation is known as a parabola**. Any equation of the

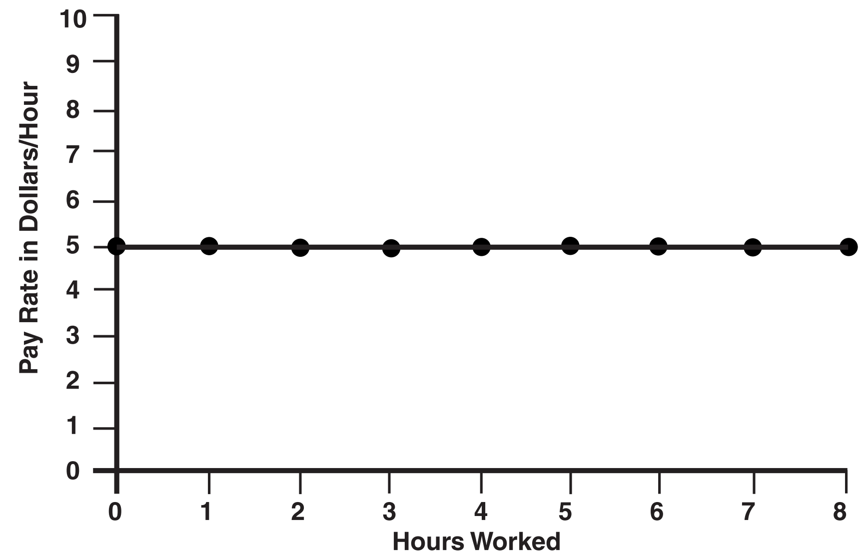

Figure 8.2a
Graph for a constant pay rate of \$5.00 per hour

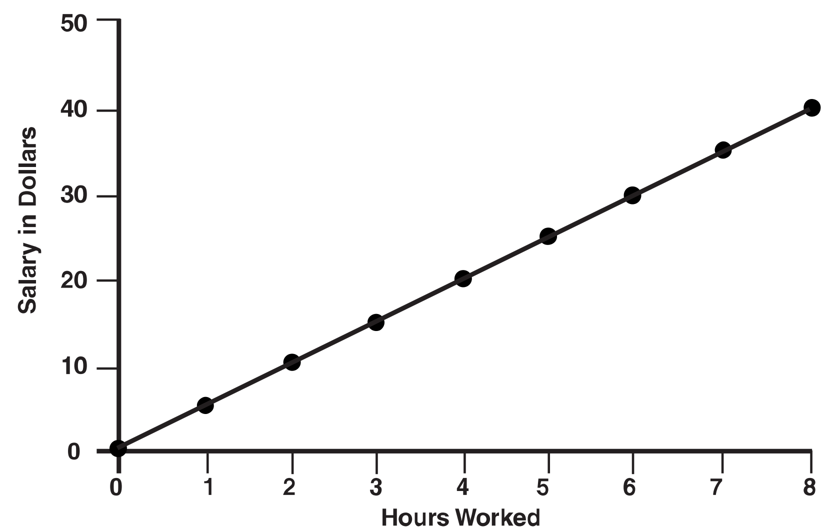

Figure 8.2b
Earnings increase linearly if the pay rate is constant

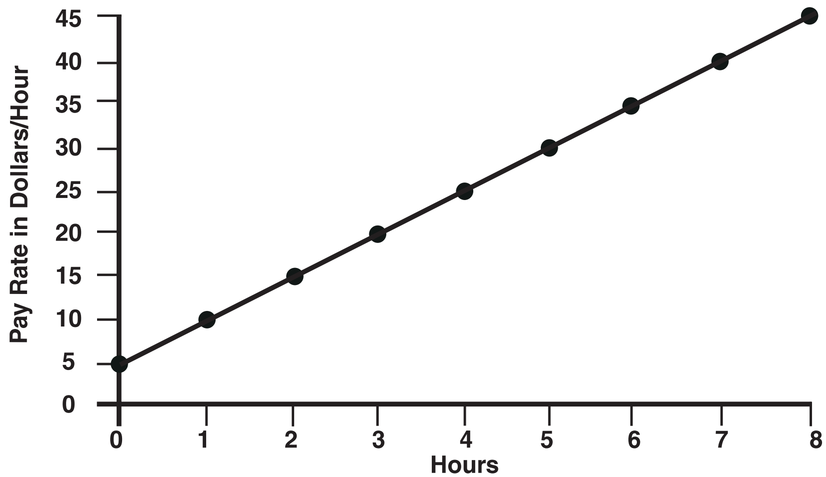

Figure 8.3a
Graph for a linear pay rate increasing $5.00 per hour each hour

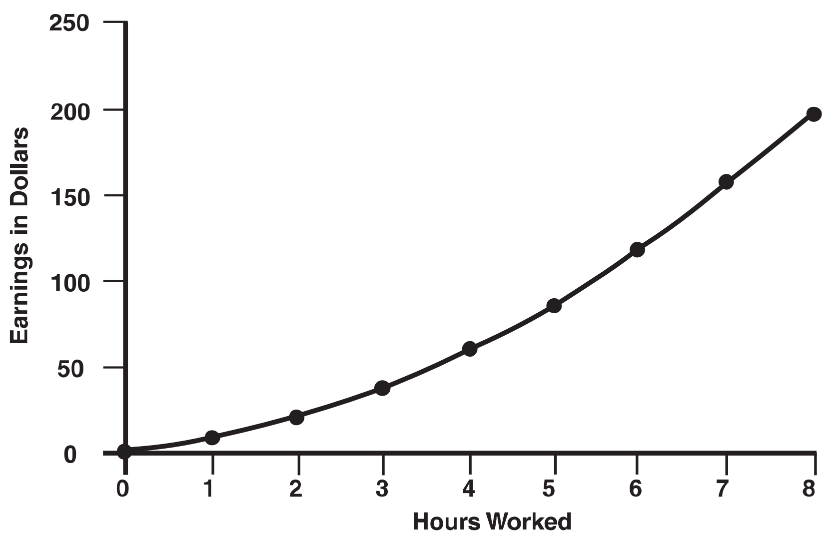

Figure 8.3b

Earnings increase quadratically if the pay rate increases linearly.

form $\mathbf{ax^2 + bx + c}$, (where a, b, and c are constants) is a parabola. In our example a = 2.5, b = 5, and c = 0 since there is no third term. As long as the constant "a" does not equal zero (even if b and c are zero) the equation remains a parabola. The graphs for the data in Table 8.1 are given in Figures 8.3a and 8.3b.

Previously, we likened our salary example to that of an accelerating car. There is a famous experiment that can also be explained using the same reasoning. In the last chapter we mentioned Galileo's experiment with falling objects. Table 8.2 shows the distance an object falls each second, and the speed it has reached at the end of each second of fall, over a time span of eight seconds. The data for this experiment is assumed to fit the condition for a "freely falling object." That is, the only force acting on the falling object is gravity—the effect of air resistance is ignored.

Table 8.2

Distance fallen and speed for a freely falling object.

Time(seconds)	Distance(feet)	Speed(ft/s)
0	0	0
1	16	32
2	64	64
3	144	96
4	256	128
5	400	160
6	576	192
7	784	224
8	1024	256

The equation that describes the distance fallen as a function of time is $d = 16t^2$ and the equation describing speed as a function of time is $s = 32t$. (See Appendix C for the derivation of $d = 16t^2$.) The first equation is a "second degree equation" (another name for a quadratic equation) where a = 16 and b = c = 0. The second equation is a

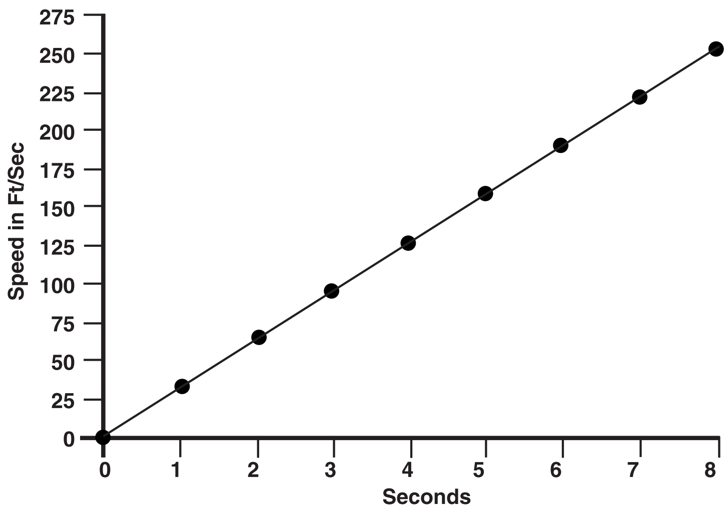

Figure 8.4a

The speed of a freely falling object increases linearly with time

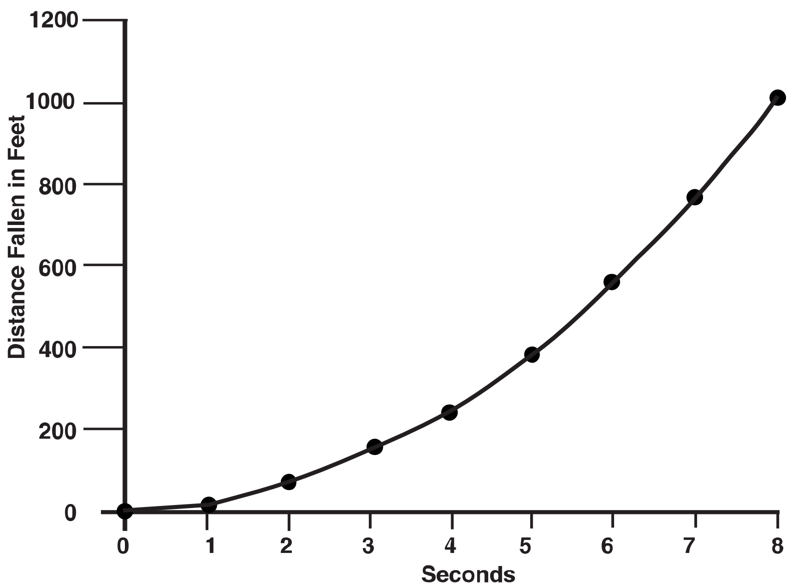

Figure 8.4b

The distance traveled by a free falling object increases quadratically with time

"first degree equation" (another name for a linear equation) where m = 32 and b = 0. Their graphs are given in Figures 8.4b and 8.4a.

Notice how fast a freely falling object is going after only a few seconds. Since gravity accelerates objects so rapidly, Galileo could not gather experimental data by allowing objects to fall freely. Instead, he ingeniously realized he could measure the nature of falling objects equally well, by using inclined planes to reduce their speed.

Figure 8.4b is a parabola. Therefore, the mathematical model that describes a freely falling object when **time and distance** are measured, is a quadratic equation (nonlinear), and takes the form $ax^2 + bx + c$ (in the particular case above, b = c = 0). However, Figure 8.4a shows that a freely falling object, when **time and speed** are measured, is a linear equation, and so must have the form y = mx + b.

How is it that an accelerating car, an increasing pay scale, and a freely falling object all have the same mathematical form with respect to distance traveled, money earned, and distance fallen? (That is, they are all quadratic equations.) The key lies with the concept of a uniformly increasing rate. The earnings for a constant pay rate were linear. When the pay rate was changed so that it uniformly increased, the earnings became quadratic. The fact that the graph for a freely falling object is quadratic, suggests that its rate of fall must also be uniformly increasing.

Indeed, after the first second of fall, experiment shows a freely falling object has a speed of 32ft/s. By the end of the second second, the object's speed has uniformly increased to 64ft/s; by the end of the third second it is 96ft/s, and so on. So just as the salary rate increases by $5.00/h each hour, the falling object's rate increases by 32ft/s each second.

There is a slight difference in the form that the quadratic equations have for these two examples. In the salary example, the quadratic equation had two terms, a quadratic term ax^2 and a linear term bx, whereas the freely falling object example had only the quadratic term ax^2. The linear term arose in the salary example because we assumed an initial pay rate of $5.00/h. That is, at the moment work began, the salary was $5.00/h. However, the freely falling object was assumed to be released from a state of rest, thus giving it an initial speed of zero feet per second. Had the object been given an initial upward or downward speed (if it had been thrown up or down, rather than released) the quadratic equation that described its motion would have had a linear term as well, which represented this initial condition.

Galileo's experiments led him to the correct conclusion that falling objects accelerate—that is, equal distances of fall do not occur with equal time increments. This is why the graph of distance versus time for a falling body is not linear. **In general, if a rate increases linearly with time, then the changing quantity will grow quadratically in time**.

The beauty in a mathematical equation is its universality. If two apparently different phenomena have the same mathematical form (linear, quadratic, etc.) then they behave similarly. Mathematics can render complex concepts into simple ones by the kind of analogies and generalizing principles discussed above. This, more than anything, is what makes the study of mathematics essential for the sciences. A secondary benefit is the possible cultivation of a broader, more holistic view for the individual—where superficial understanding is replaced by an appreciation of the underlying patterns governing nature.

Life in the Exponential Lane

The pace of life since the turn of the century has been dramatically increasing. Technological change, and with it the ability to manipulate our external environment, has profoundly altered how we live, what we need to know, and how we need to think. Since the Industrial Revolution began in the eighteenth century, the world has been experiencing rapid nonlinear growth and transformation. This is especially true of the last one hundred years. Things change so quickly today that the jobs we define ourselves by, the relationships

we form, and the values we grew up with, seem to have no permanence. A mere decade leaves one generation out of touch with another, to the extent that we don't even listen to the same music.

Ten years ago I was an exploration geophysicist with a major oil company; today I am a teacher. Ten years ago my wife was married to an air traffic controller and kept house; today she is a registered nurse working with AIDS patients. Ten years ago few households owned a microwave oven, a VCR, or personal computer; today they are commonplace. Ten years ago the clear cutting of America's ancient forests was at best an obscure issue; today it gets headlines in newspapers. Ten years ago no one worried about the deficit; today America is a debtor nation. Ten years ago it was yuppies and stock options; today it's the environment and quality of life. Ten years ago only extremists cautioned against an animal-based diet; today almost everyone is watching their cholesterol count. Ten years ago the cost of health care was not a presidential campaign issue; today the First Lady is seeking congressional and industrial backing to attack the problem. Ten years ago there were eight hundred million fewer people on the planet, today there is less food and less clean water. Ten years ago few people worried about AIDS; today many grade-schoolers are being taught about the use of condoms. Ten years ago few of us knew about the holes in the ozone layer around the polar regions; today we have advertising promoting sunscreen lotion and sunglasses that block ultraviolet radiation. Ten years ago nuclear war seemed inevitable; today the Soviet Union has collapsed and Russia is moving toward a western style economy and government. The list goes on and on.

Linearity in today's world is an archaic concept. Even the parabola is left in the dust of the crazy pace of the twentieth century. We need a new view, a different mathematical model capable of accurately describing the events unfolding today. More importantly, we need to turn the mathematics upon ourselves, to see where we have been, and where we are headed.

During previous centuries, mathematics explained the workings of the natural world—the orbits of planets, the fall of a rain drop, the propagation of sound and light. But throughout the twentieth century we have accumulated data about ourselves and the effects we are having on our planet. The interpretation of much of this data requires the use of exponential functions.

The Exponential Function

Thus far we have looked at linear and quadratic growth. Linear growth occurs when the rate of growth is constant. Quadratic growth is recognized when the rate of growth uniformly (linearly) changes. In general, this pattern could be continually extrapolated. For instance, if the pay rate grew quadratically in our salary example, the earnings would grow as a "third order equation" (that is, an equation having a cubic term, x^3 , as the highest power). Similarly, if the pay rate grew cubicly the earnings would obey a "fourth order equation."

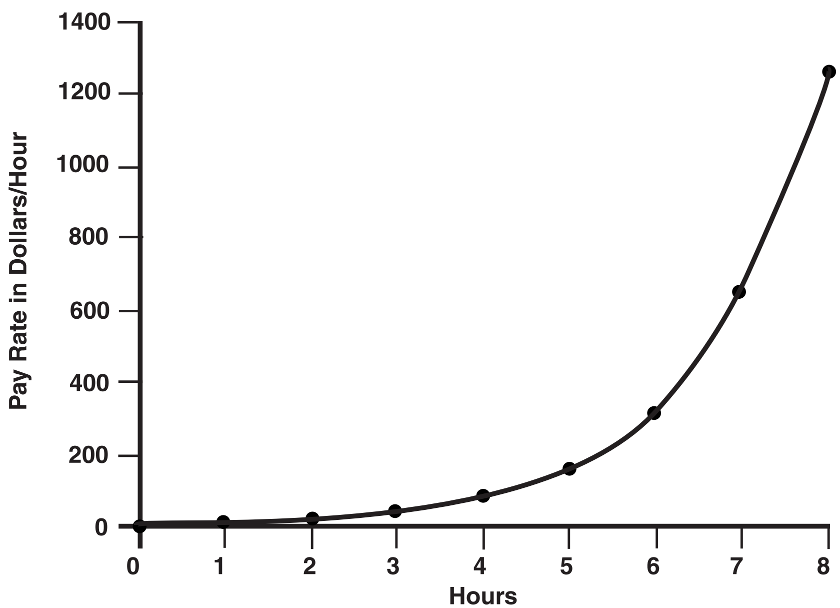

Figure 8.5a
Graph of an exponential pay rate

The pattern that emerges describes the pay rate with an "nth order equation" and the earnings with an "(n+1)th order equation," where n is any whole number. (Recall that "order" represents the highest power in the equation.) Yet regardless of how high n is permitted to be for the above kind of function (which is called a **polynomial function**), the exponential function still has the capacity for greater growth. We will see that for the exponential function, both the pay rate and earnings grow exponentially.

Using our salary example again, assume an initial pay rate of $5.00 per hour. This time, however, we let the pay rate change continuously such that the rate doubles by the end of each hour. That is, by the end of the first hour the pay rate has reached $10.00/h, by the end of the second hour, it is $20.00/h, the third hour, $40.00/h, and so forth. Table 8.3 shows the values for earnings and pay rate over an eight hour period.

Table 8.3

Earnings and pay rate for exponential growth.

Time(hours)	Earnings(dollars)	Pay Rate(dollars/h)
0	0	5
1	7.21	10
2	21.64	20
3	50.49	40
4	108.20	80
5	223.61	160
6	454.49	320
7	916.11	640
8	1839.43	1280

The function that describes the pay rate (Figure 8.5a) is $y = 5(2^x)$. That is, if $x = 6$, we have: $5(2^6) = 5(2 \times 2 \times 2 \times 2 \times 2 \times 2) = \$320/h$. The part of the equation that has a constant raised to a variable (2^x) is what governs the function and what is meant by "exponential

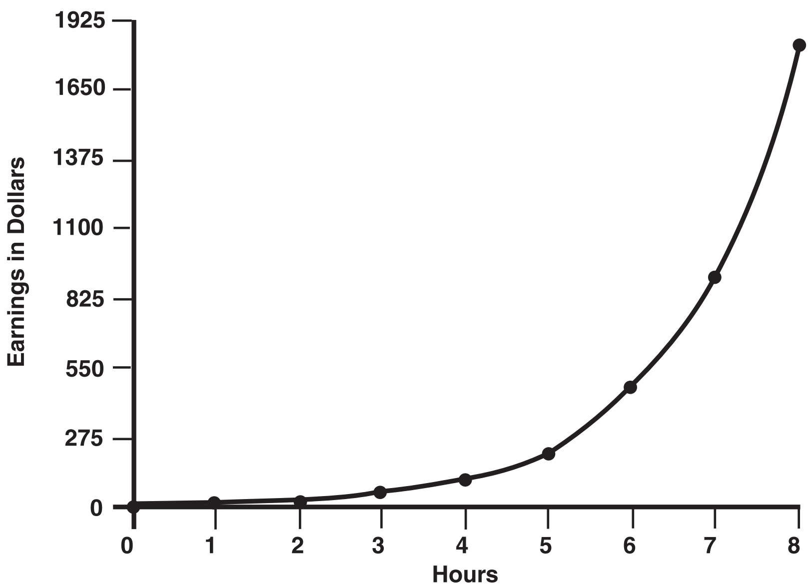

Figure 8.5b
The earnings derived from an
exponential pay rate are also exponential

behavior."

The equation for the earnings (Figure 8.5b) is $y = 7.21(2^x - 1)$. Therefore, when $x = 6$, the function is evaluated as $y = 7.21(2^6 - 1)$, which equals: $7.21(2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1)$, which is, $7.21 \times 63 =$ \$454.23. (The error in the cents (see Table 8.3) occurs because 7.21 is an approximation.)

The 7.21 in the equation above must appear quite unusual. It arises because the changing pay rate is not linear. Therefore, the "average" pay rate for any one hour interval, cannot be calculated by averaging the values at the beginning and end of each interval. This is not only true for the exponential case, but for any polynomial where the pay rate is not linear. Graphically, this makes sense because for a linear pay rate the midpoint (the average rate) for any two points on the line is also on the line. This is not generally true for a nonlinear rate. Consider a pay rate that grows quadratically to illustrate this point. The center (average value) of a straight line drawn between any two

points on the graph of a quadratic does not fall on the quadratic. Hence, the idea of averaging like this for nonlinear functions is meaningless. Such reasoning is analogous to assuming that if you travel for five minutes at thirty miles per hour and three hours at sixty miles per hour, your average speed for the total trip is forty-five miles per hour!

Once the constant 7.21 is evaluated (see Appendix C), we need only double each of the earnings and add them to the wages of the previous hour in order to find the earned money. For instance, the earnings from zero hours to the end of the first hour are \$7.21. Doubling this number gives \$14.42. This represents the money earned during the second hour of work, and therefore, the total wage from the start of the day through the second hour is \$7.21 + \$14.42 = \$21.63 (again the slight difference of a penny arises due to round-off error—see Table 8.3 page 165). Continuing in the same manner (next double \$14.42 and add this result to all the previous money earned), we would eventually generate all the values for the earnings given in Table 8.3.

In the previous example, starting with \$5.00/h put a constraint on successive pay rates while generating an exponential function. The very nature of an exponential function depends on some initial quantity. For example, when cells begin to multiply, the initial amount will determine the number at any future time. If the population in question doubles over each time increment, as in cell mitosis, the growth would follow an exponential function of the form $P = P_i(2^t)$—where P is the population at any time, P_i is the initial population, and t is time. At time $t = 0$, the population is set at P_i (since $2^0 = 1$). All future populations will be dependent upon P_i and the exponential growth of 2^t.

We often associate exponential growth with the concept of doubling. Though this is sometimes true, it is a special case of an exponential function, not the only one available. Consider the exponential function $y = 3^x$. Such a function generates the values: 1,3,9,27,81,.... Here each successive value is three times the previous

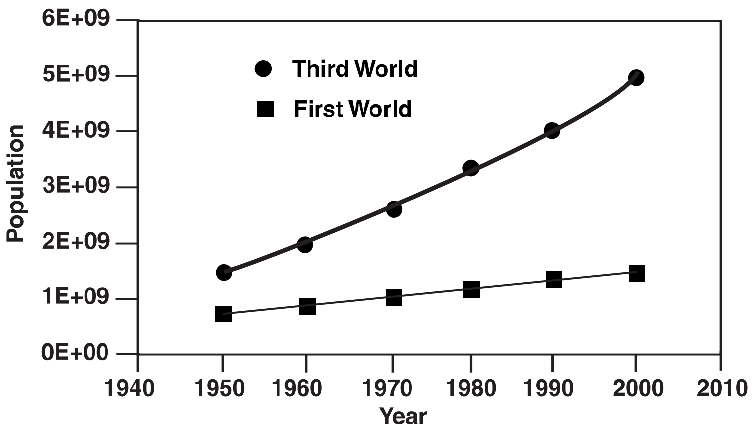

Figure 8.6

First World population growth is linear over the given time span whereas Third World population is exponential.

Source: Adapted from, *Atlas of the Environment*

one.

In general, any equation of the form $y = y_i(b^{kx})$—where y_i is a constant designating an initial amount, b is a constant representing the factor by which growth is occurring, x is a variable (usually given in units of time) and k is a constant—is an exponential function.

* * *

Figure 8.6 is a graph showing population growth for both the First and Third Worlds. (The y-axis is given in scientific notation where, for example, "6E+09" is read 6×10^9). The rate of First World growth over the domain [1950,2000] has been, and is predicted to continue growing linearly through the year 2000. By using methods from Chapter Seven, we can find an equation which describes the rate of population for this period. But the rate of Third World growth is definitely not linear; it can be shown to follow an exponential curve (see Appendix C).

* * *

Exponential functions can also be used to represent decay as well as growth; radioactive materials are a good example of this. Figure 8.7 shows the decay curve for Cesium 137, a radioactive material, which is used in cancer therapy.

The equation for Figure 8.7 is $y = y_i(2^{-t/30})$, where t is time in years, y_i the initial quantity of Cesium 137, and y the amount of material left after a time t. Note that at t = 30 years, the amount of Cesium 137 remaining is half of the initial value (recall $2^{-1} = 1/2$). Thirty years is therefore the "half-life" of Cesium 137. The problem with many radioactive materials is that their half-life can be centuries. Some of the radioactive waste we are creating today will still be hazardous hundreds of years from now.

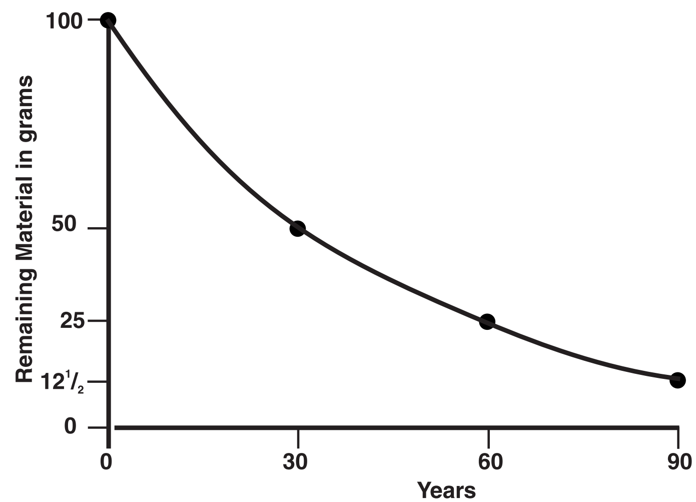

Figure 8.7

An example of exponential decay—Radioactive Cesium 137 with an intial amount of 100 grams.

Figure 8.8 is a graph showing the three functions we have discussed so far—linear, quadratic, and exponential. There is, at times,

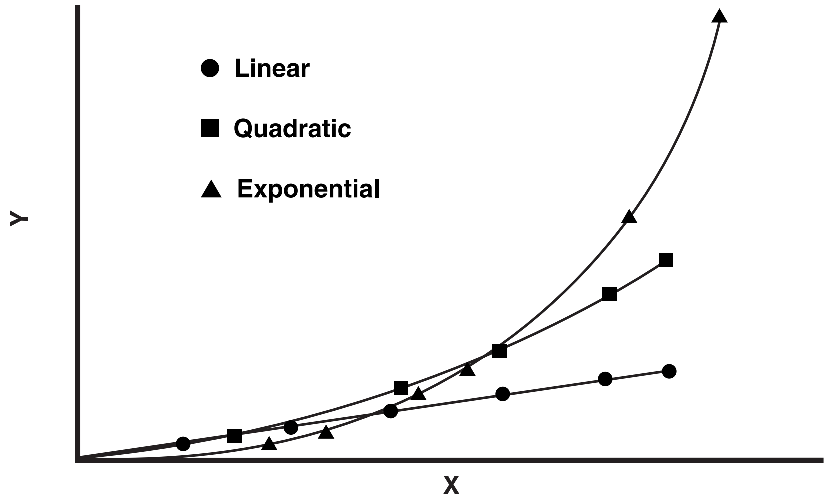

Figure 8.8
Comparison of linear, quadratic, and exponential growth

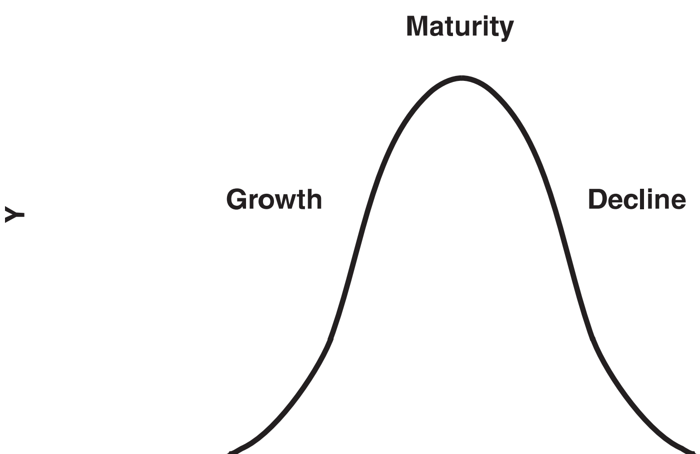

Figure 8.9
A bell-shaped curve approximates a life-cycle curve

confusion over the fact that quadratic functions are not exponential functions. A quadratic function has exponents, as do all polynomial functions, but the exponents are constants. Exponential functions

have exponents which are variables, and therefore grow or decay exponentially.

Notice how all three functions increase without bound, yet nothing in nature increases forever. Most things grow, mature, and die. This is true of organic life-cycles, or the life-cycles of resource use, technological innovation, or the creative output of scientists and artists.[2] Consider Figure 8.9 which is commonly referred to as a bell-shaped curve.

We can use this curve to make a point regarding growth, maturity, and decline. Finite "resources" like oil, ancient forests, and whales, are subject to increasing use until their consumption is maximized. If the demand continues, and the resource is still exploited, production will decline and the resource will be exhausted. Creative output follows a similar path. There is usually a start-up period, followed by increasing output that reaches a ceiling of maximum productivity. Afterwards, output tapers off, and a complete cessation of activity occurs with death. Many new products in the marketplace follow a similar life-cycle.

Fads fit exceptionally well into this model also. The hula-hoops of the 1950s and 1960s, and the mood rings, pet rocks, and leisure suits of the 1970s are a few examples of such fads.

Fads spread like diseases, quite literally. There is usually an initial slow-to-modest growth period until the item "catches-on," after which growth rapidly takes off, until reaching some maximum level. Soon after this, interest in the item wanes and the number sold begins to decline quickly, until sales diminish to some nominal level. Items that have longevity, like television sets, follow the same beginning but do not fade out as dramatically as do fad items.

Within a year or two after telephone answering machines went on the market, everyone I knew purchased one. The same has been true of VCRs and computers. Each item has a respective "niche" to fill. The buying frenzy for all new innovations slows down when everyone who is going to buy the item already has. Afterwards sales level off, fueled by replacement items and the next generation who purchases their own.

After the item fills its niche, the rapid growth it enjoyed is over; though it may continue growing as the "need" for the item persists with increasing population. But given enough time, any item will eventually become extinct. Phonograph records are a good example of this. The invention of the phonograph created a market for records that spanned about seventy years. But the compact disc, a more successful competitor, overtook the niche of the vinyl record. Records themselves have been subjected to evolutionary changes, where one form died out in place of another. The first disk records were made from metal and sprayed with vinyl. Later "generations" were made with 100 percent vinyl, and continued to became more flexible. But it has been only within the last few years that records as a "species" have stopped reproducing (numerically) because of competition from the more successful compact disk.

The bell-shaped curve in Figure 8.9 can be used to generate a logistic function, also called an S-curve. **An S-curve shows the total accumulation from the life-cycle curve and therefore never diminishes**. (Bell-shaped curves or "normal" distributions are not exactly the same as a true life-cycle curve, but the difference for our purposes is negligible.)

Theodore Modis has a section in his book *Predictions* entitled, "Man-made Dinosaurs," where he discusses the life-cycles of such things as supertankers, Gothic cathedrals, and particle accelerators. Particle accelerators are used to investigate the creation and decay of subatomic particles. Each of these innovations has completed its life-cycle; the only exception possibly being the particle accelerator wherein the potential for one or two more still exists.[3]

Supertankers will serve as a representative example for explaining Modis's "Man-made Dinosaurs." During the 1970s, ships called supertankers were constructed with a carrying capacity of more than three hundred thousand tons. But over a period of ten years or so, competition with smaller ships showed they were not capable of holding their own in the marketplace. The 1980s and 1990s have

witnessed them steadily being replaced by the smaller vessels. It will not be many more years before the last of the ships are decommissioned.[4]

Figure 8.10a shows the life-cycle curve for supertankers and Figure 8.10b shows its corresponding S-curve.[5]

At the beginning of the innovation's life, growth was modest and so the S-curve rose slowly. As the innovation caught on, the **cumulative number** of ships increased rapidly and the S-curve rose more sharply. In the final part of its life-cycle, less and less were produced and the corresponding growth of its S-curve flattened out. The final part of the curve has approached a limiting value which can be thought of as the niche capacity for the innovation.

S-curves are another example of an exponential function. One way to represent them mathematically is with the equation $y = M/[1+A(2^{-at})]$, where M is a constant denoting the niche capacity, A and a are constants found from the initial conditions of the problem, and t is time. S-curves are used to explore social and commercial trends. These trends are often helpful in analyzing human behavior on a macroscopic scale. They also lend insight into the productivity of individuals and their ultimate potential. The idea of personalizing S-curves to individuals is really the work of Cesare Marchetti. As Modis explains in his book:

> Cesare Marchetti was the first to associate the evolution of a person's creativity and productivity with natural growth. He assumed that a work of art or science is the final expression of a 'pulse of action' that originates somewhere in the depths of the brain and works its way through all intermediate stages to produce a creation. He then studied the number of these creations over time and found that its growth follows S-curves. Each curve presupposed a final target, a niche size he called a perceived target, since competition may prevent one from reaching it. He then proceeded to study hundreds of well-documented artists and scientists. In each case he took the total number of creations known for each of these people, graphed them over time, and determined the S-curve that would best connect these data points. He found that most people died close to

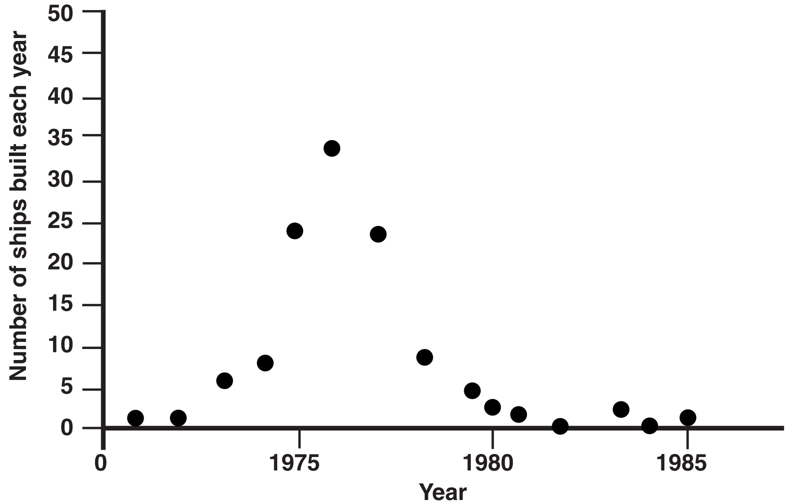

Figure 8.10a

The rate of growth for supertankers approximates a bell-shaped curve

Source: Adapted from *Predictions*

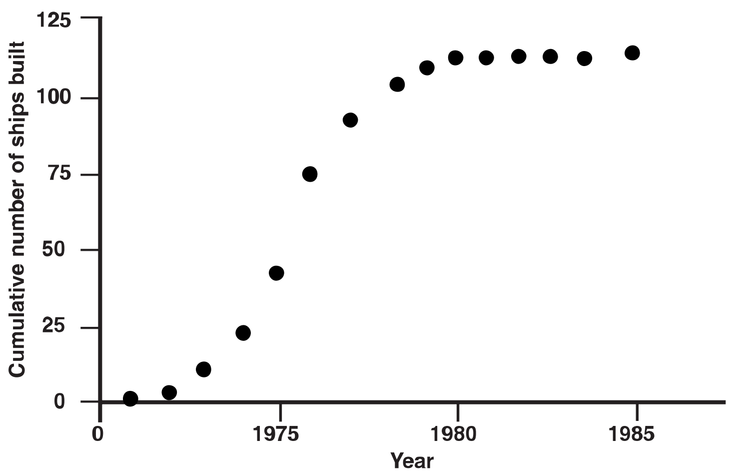

Figure 8.10b

The cumulative growth of supertankers follows an S-curve

Source: Adapted from *Predictions*

having realized their perceived potential.[6]

Marchetti believes that most people develop somewhere between 90 to 100 percent of their potential, and even those who die young (like Mozart who died at thirty-five), do so because they have nothing left to say.[7] There is the implication in both the work of Modis and Marchetti that "temporal age" may be only one form of aging, and we need also to consider "creative age." From this point of view, Mozart died young chronologically but old creatively. Modis seems to accept this though his own research allows for accidents—as in the case of the poet Percy Bysshe Shelley, who died at age thirty after having completed only half of his S-curve. Modis also makes an argument that suicide may play a role in a person's life when they are creatively exhausted.

There is a ring of predestination in both the work of Modis and Marchetti that some may find intellectually troubling. Their work may make more sense if we compare their views on creative output to biological output.

Genetically, every living thing from an oak tree to a human being is programmed to reach its optimum height, given the proper environment in which to do so. By extending this idea to creativity, Modis and Marchetti are showing a psychic analogue to biological programming. Such an idea is not so unreasonable. We may, after all, intuitively realize this each time we make reference to an individual not fulfilling their potential. Indeed, from an educational point of view, it supports the value of optimizing a child's home and school environment so that nurture may work hand in hand with nature.

In a sense we have come full cycle. How much mathematics we really do need to know, ultimately depends upon the life-cycle of the society we inhabit. Collectively, we are each part of the ebb and flow of the events that shape our society. Any means available to us to gain a greater understanding of ourselves is beneficial.

Mathematics is an essential tool in this quest given the present evolution of our culture. It is an important piece in the complex tapestry of the civilization that each of us has inherited. We owe it to ourselves and our children to insure that no subject of human knowledge is accessible only to a chosen few.