CHAPTER FIVE

Nuts and Bolts

Creating a clear and concise notational system is the first step in formulating more useful mathematics. In sharp contrast to this are the number systems of ancient societies which were ill-suited for even simple multiplication and division. A good notational system can be compared to a fine musical instrument. Both the musical instrument and the notational system hold incredible potential and yield amazing results in the right hands. The key in both instances is exploration and imagination.

Playing with Pieces—A Look at Fractions

Below are four interpretations of fractions:

1. A fraction represents a part of something, or a whole plus a part of something. For example, ¾ is a part of something while 4/3 is a whole plus a part.
2. A fraction represents division. Since we freely used a / to represent ÷ in Chapter Four this must come as no surprise. For example, ¾ = .75; the denominator is divided into the numerator (bottom number into top number) to find the "decimal equivalent" for a given fraction.
3. A fraction represents the product of a whole number and a unit fraction. Three-fourths is equivalent to 3 × ¼ (¾ = 3 × ¼).
4. Any fraction can be thought of as a ratio. (¾ can mean, 3 compared to 4, or 3:4.)

Of all the different ways to write fractions, the third one in the list on the previous page is the most difficult for people to understand. Ironically, it also offers one of the most insightful views of fractions.

Any fraction can be expressed as a whole number times a unit fraction—where a unit fraction has 1 in the numerator such as $1/2$ or $1/5$. The fraction $3/4$ can be rewritten as the whole number 3 times the unit fraction $1/4$ ($3 \times 1/4$). Think about what $3/4$ represents—three one-fourths. Similarly, 5/6 represents five one-sixths. What does three twos mean? You could write this as 2,2,2 or $2 + 2 + 2$ or 2×3. The harder to understand fraction concept works the same way. Three-fourths can be thought of as $1/4$,$1/4$,$1/4$ or $1/4 + 1/4 + 1/4$ or 3 times $1/4$ ($3 \times 1/4$). Furthermore, if the phrase, "three twos" is taken to mean, three sets of two, or two sets of three, then $3 \times 1/4$ can be interpreted as 3 sets of $1/4$ or $1/4$ of a set of 3. But notice what happens when we word it this last way ($1/4$ of a set of 3). One-fourth of a set of three, is some part of three. We're taking a part of something, but that sounds like division. But we're multiplying! We know we are multiplying because if we say it in reverse, three sets of one-fourth, we see the additive principle we associate with multiplication. Since $3 \times 1/4$ is no different than $1/4 \times 3$ (just as $2 \times 3 = 3 \times 2$) it must mean both statements are equivalent. Therefore, multiplication by fractions is equivalent to division. If we wished to, we could eliminate the traditional concept of division and replace it with multiplication (though I don't advise it). Perhaps we should also replace the word "multiplication" because it gives the misleading notion of a purely "increasing" process, which is certainly not true when multiplying with fractions.

Decomposing a fraction into a whole number times a unit fraction also offers insight for the rule governing multiplication with fractions. Since 3 is the same as $3/1$, then $3 \times 1/4$ is the same as $3/1 \times 1/4$ which we know is $3/4$. The most immediate way to arrive at this answer is to multiply across, that is, 3×1 over 1×4, which is $3/4$. The same process is true for any numbers, for example:

$$3/8 \times 5/7 = (3\times5)/(8\times7) = 15/56$$

Some of the people who missed the multiplication problem with fractions on my survey (see Chapter Two) tried to multiply the numerators and add the denominators. Such attempts show no understanding of the concept of multiplication with fractions. Let's work with the previous problem to show the rule for multiplication in a slightly different way than we did using $3/4$.

We can write $3/8 \times 5/7$ in various ways by using unit fractions, and the fact that the order of multiplication is irrelevant. We have:

$(3 \times 1/8) \times (5 \times 1/7) = (3 \times 5) \times (1/8 \times 1/7)$, which immediately shows that the numbers in the numerator must be multiplied together. A similar argument for multiplying 8×7 in the denominator can be used as was done for $3/1 \times 1/4$. It is better though, to try and develop a "feel" for why this operation yields reasonable results. Remember, $1/8 \times 1/7$ can be thought of as a one-eighth part of one-seventh. Visualize taking a seventh of a piece of pie, and then taking one-eighth of this small piece. In other words: What size piece of pie will we have if a seventh of a piece of pie is cut into eight pieces? Since there are seven, one-seventh pieces in a pie, and each seventh is cut into eight pieces this makes for a total of 8×7 tiny pieces. Therefore, if the total pie is cut into 56 pieces, than one piece is $1/56$ of the pie.

This is a tremendous amount of work to try to show the logic (not proof) of fraction multiplication, namely $a/b \times c/d = (a \times c)/(b \times d)$. The real value in all this arithmetic is to make the reader aware of how to manipulate fractions in this manner.

We could also view the problem $3/8 \times 5/7$ as three-eighths of a set of five-sevenths. Visualize what this statement is saying by approximating. We are taking nearly one-half ($3/8$) of nearly three-quarters ($5/7$). Which is like taking one-half of 75 cents or about 38 cents or $38/100$ of a dollar. How close is the approximation to the actual answer? The decimal equivalent of $15/56$ is about .27 (divide 56 into 15) or $27/100$. Even though our quick approximation is off a fair amount, our answer is not ridiculous. This is important because it lends support to our view of multiplication with fractions as taking a part of something, or in this case, taking a part of a part.

One of the major stumbling blocks for students enrolled in algebra (perhaps a better name for algebra is symbolic arithmetic) is that they don't understand basic operations with fractions. Many algebraic manipulations depend on the general statement $a/b = a \times 1/b$. Yet students as far along as trigonometry are often unable to apply this concept.

In Chapter One I discussed division of fractions. Recall that none of the middle school teachers I asked understood why inverting and multiplying gave the correct answer for dividing two fractions. The rationale for this rule hinges on using the number 1 in a creative way. Any number (other than zero) divided by itself is equal to one. That is: $\frac{5}{5}$ or $\frac{.7}{.7}$ or $(\frac{3}{4}) / (\frac{3}{4})$ are all 1. Furthermore, multiplication by one leaves a number unchanged. That is: 5×1, $.7 \times 1$, or $\frac{3}{4} \times 1$, all remain 5, .7, and $\frac{3}{4}$.

Now consider the division problem $\frac{3}{8} \div \frac{5}{7}$. This can be rewritten as $(\frac{3}{8}) / (\frac{5}{7})$. If we multiply the fraction $\frac{3}{8}$ over $\frac{5}{7}$ by 1, we still have the same fraction. But what if we express "1" as $\frac{7}{5}$ over $\frac{7}{5}$ and multiply this by $\frac{3}{8}$ over $\frac{5}{7}$. We have: $(\frac{3}{8}) / (\frac{5}{7}) \times (\frac{7}{5}) / (\frac{7}{5})$, which is, $(\frac{3}{8} \times \frac{7}{5})/(\frac{5}{7} \times \frac{7}{5})$. The denominator multiplies out to 1, and we are left with the numerator which is itself the multiplication problem $\frac{3}{8} \times \frac{7}{5}$. This is what we have if we invert and multiply. The rule "invert and multiply" comes from answering the question: By what number must the bottom fraction be multiplied to reduce to one? This number is the "reverse" (the proper term is inverse or reciprocal) of the bottom fraction. The number 1 is then expressed as this reversed number divided by itself, and multiplied with the original problem. Hence, division is turned into multiplication, by multiplication with "1".

In Chapter Three, I briefly discussed addition of fractions. The important concept here is to make sure likes are added to likes. In order to add $\frac{1}{3}$ and $\frac{1}{5}$ we must have a "standard" by which both fractions can be compared. The usual rule is to find the "least common denominator" for both 3 and 5. This is the smallest number both 3 and 5 will evenly divide into, which in this case is 15. We can re-

express $\frac{1}{3}$ and $\frac{1}{5}$ in terms of fifteenths by again imaginatively using the number 1. If $\frac{1}{3}$ is multiplied with $\frac{5}{5}$, and $\frac{1}{5}$ is multiplied with $\frac{3}{3}$, the results are $\frac{5}{15}$ and $\frac{3}{15}$ which totals $\frac{8}{15}$. One difficulty with adding fractions is that the least common denominator is not always easy for people to discern. In cases where the least common denominator is not readily apparent, it can easily be avoided. The only important consideration for adding fractions is to make sure all the denominators are the same number. What if we had completely ignored the question of a least common denominator when we were adding $\frac{1}{3}$ and $\frac{1}{5}$ and just quickly sought any number that would make both denominators equal. Let's say we quickly notice that 3×10 and 5×6 are both 30 and decide to transform both denominators into 30. To do so, we multiply $\frac{1}{3}$ with $\frac{10}{10}$ and $1/5$ with $\frac{6}{6}$. This gives $\frac{10}{30}$ and $\frac{6}{30}$ which added together gives $\frac{16}{30}$. But $\frac{16}{30}$ can be reduced since $16 = 8 \times 2$ and $30 = 15 \times 2$. Therefore, $(8 \times 2)/(15 \times 2) = 8/15 \times 2/2$ which is $8/15$. It may, at times, be easier to forget about finding a least common denominator and use any number that will give equal denominators. Then, after the problem is worked, it can be put in lowest terms.

* * *

Fractions are not as simple, nor are the "rules" governing their operations, as obvious as many authors of mathematics books would have us believe. Operations with whole numbers border on the intuitive for most people, but the same cannot be said for fractions. It is for this reason that ancient societies, as well as current college graduates, have had difficulties with the subject.

Percentage

One of the more startling revelations of my study was that one out of four persons could not compute eight percent of six dollars, yet percentages are used in most professions and in many daily transactions. They are used to compute federal, state, and city taxes. They are present when buying, and selling, and tipping. Merchants use them to lure customers to buy their wares by announcing "40 percent off the retail price." Batting averages, bank loans, salary increases, and grades on tests are all stated with percentages. Everyone should know what percentages represent and how to work with them.

Percentages, when one is comfortable with them, immediately provide for a sense of proportion. Which is easier to visualize, 5⁄7 of a pie or a little over 71 percent? Percentages are based on one-hundred. So 71 percent is the same as 71 pennies out of one hundred pennies, which is more easily understood than 5⁄7 of one hundred pennies.

For the purpose of computing with money, 8 percent is seen as 8 cents on every dollar (eight pennies out of one hundred pennies). Eight percent of two dollars is 16 cents because each dollar represents 8 cents. Therefore, 8 percent of $6.00 is 6 × 8¢ = 48¢. Similarly, 15 percent is 15 cents on a dollar. Fifteen percent of $6.00 is 6 × 15¢ = 90¢. To find the 8 percent sales tax on $6.50 only requires determining 8 percent of 50¢, since 8 percent of 6 dollars is already known. The appropriate question is: If 8 percent means 8 cents for each dollar, how many cents for only half of a dollar? Since 50¢ is half of one dollar (100 pennies) then the tax should be half as great, or 4¢. Likewise, since there are four 25 cent units in a dollar, each 25 cents must have 2¢ of the eight cent burden upon it.

So 8 percent of $6.50 is: (6 × 8¢) + 4¢ = 52¢
Similarly, 8 percent of $6.25 is: (6 × 8¢) + 2¢ = 50¢
What is 8 percent of $6.75?

The above line of reasoning will always work, but it is slow and cumbersome. A more illuminating pattern is needed.

We already have determined that 8 percent of $6.50 is 52 cents. Notice what happens when $6.50 is multiplied by 8:

$6.50 × 8 = $52.00

Obviously, fifty-two dollars cannot be the answer, but the fifty-two is correct; it's only a matter of where the decimal point is placed. If the decimal point were moved two places to the left, this would give $.52 (fifty-two cents). But moving a decimal point two places to the left corresponds with division by 100, or multiplication by 1⁄100. If the problem had originally been written as $6.50 × 8 × 1⁄100 or $6.50 × .08, fifty-two cents would immediately follow.

The above suggests a rule: **To compute using a percentage, drop the percent sign (%) from the number, then place the number over 100, express decimally and multiply**. This "rule" should come as no surprise, since the concept of percentage is based upon 100.

Consider the following example: Find 15 percent of $56.87.

15% = 15⁄100 = .15
$56.87 × .15 = $8.53

What if, however, we are given any fraction and asked to express it as an equivalent percentage? Consider the fraction ¾. If the 4 is divided into the 3 the result is .75, so ¾ and .75 mean the same thing. If the rule for turning a percent into a decimal is internally consistent, then the reverse—turning a decimal into a percent should be: **Multiply the decimal by 100 and then put in the percent sign**.

Thus, .75 x 100 = 75%

This implies that fractions, which are convertible to decimals (by dividing the denominator into the numerator) must also represent percentages. It must be that 3/4 is equivalent to 75 percent. Any fraction is therefore equivalent to a percentage, and hence, expressible with respect to one-hundred.

Find the equivalent percentage for 60⁄250, and rewrite it with respect to one-hundred.

60⁄250 = .24 (by dividing 250 into 60)

Converting to a percentage: .24 × 100 = 24%

Since all percentages are based on a hundred we have:

$60/250 = 24/100$.

This tells us that 60 is related to 250 the same way 24 is related to 100. So a pie cut into 250 pieces, will be about one-fourth consumed when 60 pieces have been eaten.

As a final example, we will look at the ratio of the national debt to the Gross National Product (GNP). Earlier it was stated that the national debt has increased from $75 million to $4 trillion since the founding of the United States. Though the national debt has apparently grown substantially over the years, the dollar figures by themselves are not very revealing. Without a more objective measure, there is no way to understand what the changing debt represents. Though many factors should be considered to develop an accurate picture of America's rising debt, considerable insight can still be glimpsed by the simple ratio of debt to GNP.

A ratio (fraction) can be regarded as a comparison of top number to bottom number. Calculating this ratio for 1981 and 1992 will give an absolute measure of how our debt has risen with respect to our production over this eleven year period. (The larger the span of time used, the less meaningful the comparison becomes, since so many more variables—energy requirements, education, recreation, etc.—are introduced. Hence, a comparison of this ratio over a two hundred year period would be rather meaningless.)

This ratio (percentage) went from 33.3% in 1981 to 67.2% in 1992.[1] In absolute terms, this means that for each one-hundred 1981 dollars of the GNP, $33.30 was already committed to debt. Eleven years later, out of every one-hundred 1992 dollars, $67.20 was debted.

Just as an individual has less money for daily cost of living expenses when burdened with debt, so too does government.

The following examples are provided to reinforce concepts involving percentages.

Examples

1. If you earned $30,000 last year and paid $8,400 in taxes, what percent of your income did you give Uncle Sam?

In computing percentages, it is always important to ask yourself: What is being compared to what? Or: What is the part and what is the whole? In this example the whole is $30,000, so the ratio is: $8400/30000 = .28$, $.28 \times 100 = 28\%$.

2. Out of every one-hundred dollars earned in the above example, how much is left to the taxpayer?

Since 28% means 28 out of one-hundred, what remains is 72 dollars, per each hundred earned.

3. You pay $50.00 for an item and sell it for $125.00, what was your profit in dollars and what percentage is this?

In dollars: $125.00 − $50.00 = $75.00 profit.

Percentage: $75.00/$50.00 = 1.50, $1.50 \times 100 = 150\%$. (Percentage of profit is defined as: (profit in dollars/cost to seller) times 100.) See Appendix A for an alternative definition.

Example number 3 should make it clear that fractions, ratios, and percentages can be greater than 1, or equivalently greater than 100 percent. It can be misleading to think part/whole when the "part" can at times be larger than the "whole". To avoid this confusion, clarify exactly **what** is being referenced to **what**. Think more along the lines of **compared amount/base amount**. In example three, the profit was compared to the cost. There is no reason it could not have been reversed.

4. It was stated previously that 67.2 percent of the GNP in 1992 was tied to the debt. Approximate the GNP for 1992.

Since we wish an approximation, we can take advantage of the fact that 67.2% is very close to 66.66 … % which is an easy fraction to work with, namely ⅔, which yields a repeating decimal, (66.66 … % = 66.66 … /100 = ⅔.) Therefore, the ratio of debt to GNP is about 2:3. Since the debt is approximately $4 trillion we write:

4/? = ⅔, by observation, 4⁄6 reduces to ⅔, so the GNP must have been approximately $6 trillion for 1992.

Lastly, we show how the result on page 70 in Chapter Four was calculated.

> 5.) How much would a person have to save out of an annual salary of $25,000, if the proportion was to be the same as for a $10 million savings out of $327 billion?

Since $10 million out of $327 billion was saved, we have: $10,000,000/$327,000,000,000 = 1/32,700 = .00003058. Therefore, $25,000 × .00003058 = 76¢.

Powers of Ten and Scientific Notation

In the past, RBNs were cumbersome to work with because of their size. Multiplication and division with such numbers were slow and tedious. Today, calculators manipulate RBNs with ease, but in doing so, often resort to a mathematical shorthand called scientific notation. The study of scientific notation is dependent on yet something more fundamental, called "powers of ten" notation.

Numbers such as 10, 100, 1000, 10000,. . ., can all be expressed as products of ten times itself. That is, ten is simply 10, one-hundred is 10 × 10, one-thousand is 10 × 10 × 10, ten-thousand is 10 × 10 × 10 × 10, and so on. Instead of writing all these zeros a shorthand notation is employed:

$100 = 10^2$
$1{,}000 = 10^3$
$10{,}000 = 10^4$
$100{,}000 = 10^5$

$1{,}000{,}000 = 10^6$
$1{,}000{,}000{,}000 = 10^9$
$1{,}000{,}000{,}000{,}000 = 10^{12}$

Note that the numeral written above the zero of ten (called an exponent) corresponds to the number of zeros in the original number.

A million, billion, and trillion can now be expressed as:

10^6 = one million
10^9 = one billion
10^{12} = one trillion

In general, each time the exponent increases by three, a new name is given for the number, for example: 10^{15} is called a quadrillion and 10^{18} is called a quintillion.

The Universal Almanac gives names for numbers up to 10^{33} (a decillion) and then jumps to 10^{100} which is given the name googol and finally $10^{10^{100}}$ which is called a googolplex. Often, metric prefixes are assigned to these numbers. The term megabyte, used with computers, means 10^6 (a million) bytes. (A byte represents the information from eight "data lines" that are represented by a series of ones and zeros, called bits.) Other common metric prefixes are: giga is 10^9 (a billion), tera is 10^{12} (a trillion), peta is 10^{15} (a quadrillion), and exa is 10^{18} (a quintillion). As stated earlier, numbers beyond a trillion are not generally encountered, except in scientific literature, and those that are, usually are stated only as a power of ten without regard for prefixes.

Numbers such as 10^2 and 10^3 are sometimes referred to as ten squared and ten cubed, respectively. Larger numbers, such as 10^4 are generally read as "ten to the fourth" or "ten raised to the fourth power."

Some of the numbers quoted in Chapter Four are restated below:

1. The four trillion dollar deficit: $\$4 \times 10^{12}$.

2. The eight hundred seventeen billion dollars for health care: $\$817 \times 10^9$.
3. The present world population of five billion three hundred million people on Earth: 5.3×10^9.
4. The four hundred fifty million motor vehicles: 450×10^6.

There is only a small step required in going from powers of ten to scientific notation: **Numbers written in scientific notation are expressed as a number between one and ten which is multiplied by the appropriate power of ten**. In the examples given above, only the first and third are already in scientific notation. In order to see how to express numbers in scientific notation, we must first explore some of the properties involving powers of ten notation.

We begin by multiplying 100×1000; this yields 100,000. Rewriting the problem in powers of ten gives: $10^2 \times 10^3$. Since one-hundred thousand can also be expressed as 10^5 we can then write $10^2 \times 10^3 = 10^5$. (Remember that the exponent represents the number of zeros.)

How about $10^3 \times 10^4$? This is $1,000 \times 10,000$ which equals 10,000,000. So $10^3 \times 10^4$ must be 10^7. Much of the success (and pleasure) in learning mathematics rests on seeing patterns. Consider the two examples just presented:

$$10^2 \times 10^3 = 10^5$$
$$10^3 \times 10^4 = 10^7$$

Do you see any possible relationship with the exponents in each case?

A good guess would be: When multiplying powers of ten, add the exponents of the two tens on the left and use this sum for the exponent on the right. We can test this idea by taking other examples. If our guess is correct, then $10^6 \times 10^6$ should be 10^{12} (since $6 + 6 = 12$). Recall that 10^6 is a million and 10^{12} is a trillion (see above list). Does one million times one million equal a trillion? It certainly does.

Therefore, to answer a question asked in Chapter Four, it takes a million millions to equal one trillion. A trillion is truly a **R**eally **B**ig **N**umber!

Generalizing these results for any exponents a and b gives: $10^a \times 10^b = 10^{a+b}$.

We are now ready to tackle scientific notation. Consider the dollar amount cited for health care on page 64. In order to express 817 as a number between one and ten (which, as cited earlier, is a requirement for scientific notation) we write it as 8.17×10^2. **The exponent of ten, is now more generally understood as representing the number of places the decimal point is moved**. Since 817 has an implicit decimal point after the 7, it is rewritten as 8.17 by moving the decimal point two places to the left, and then compensating by multiplying this smaller number with ten raised to the appropriate power ($8.17 \times 10^2 = 8.17 \times 100 = 817$.) We can now express 817×10^9 as $8.17 \times 10^2 \times 10^9 = 8.17 \times 10^{11}$ (since we know $10^2 \times 10^9 = 10^{11}$). Similarly, in example four, 450×10^6, can be restated in scientific notation as $4.5 \times 10^2 \times 10^6$ which equals 4.5×10^8.

Scientific Notation has yet another advantage. The value for the tobacco crop given in the last chapter was \$1,929,763,000. There is a degree of ambiguity concerning the three zeros on the end of this number. Are we to assume this was exactly the value of the crop, or that the figure was rounded to the nearest thousand dollars? Considering that numbers in the real world don't usually work out so evenly, the dollar value cited is probably an approximation valid to the nearest thousand dollars. To show this using scientific notation, the number is written as $\$1.929763 \times 10^9$, where the three zeros are not included. On the other hand, if the value is exact to the dollar, the figure is written as $\$1.929763000 \times 10^9$. This tells the reader the zeros are, to use the correct phrase, "significant figures", rather than place holders used for an approximation.

Try this example: On average, each person's share of the national debt is \$16,000. If there are 250 million people in the United States what is the total debt owed?

Using scientific notation we have: $\$1.6 \times 10^4$ per person $\times 2.5 \times 10^8$ people. This can be rewritten as $(\$1.6 \times 2.5) \times (10^4 \times 10^8)$ which equals $\$4.0 \times 10^{12}$. The reverse of this question was asked on page 70 and the answer should have been \$16,000.

RBNs are divided as well as multiplied. We therefore need to investigate the properties of our notational system for division.

Consider the two expressions 10,000/100, and 100,000,000/1,000, which are equal to 100 and 100,000, respectively. Rewriting with powers of ten gives:

$$10^4/10^2 = 10^2$$
$$10^8/10^3 = 10^5$$

Can a simplifying rule (a recognizable pattern) for division be postulated?

Notice that if the exponent of the bottom number is subtracted from the exponent in the top number, this yields the exponent in the answer. Other examples would continue to support this process. Generalizing this pattern for any exponents a and b gives: $10^a/10^b = 10^{a-b}$

In 1950, world oil production was roughly ten million barrels per day. The United States, alone, uses more than this today. If there are, to the nearest power of ten, approximately one hundred thousand seconds in a day, about how many barrels of oil were produced in 1950 per second, on average?

10^7 barrels/10^5 seconds = 10^2 barrels/second or
100 barrels/second

Let's make the previous problem regarding the amount of money owed by each person on the national debt a division problem. This will now be the same problem asked on page 70. Using scientific notation the problem is written:

$(\$4 \times 10^{12})/(2.5 \times 10^8)$ people = $(4/2.5) \times (10^{12}/10^8) = \$1.6 \times 10^4$ per person, where 4/2.5 is treated as one calculation and $10^{12}/10^8$ as another.

I have purposely not yet stated values for 10^0 or 10^1, since I have found they often confuse students. We will begin with 10^1 first. Since $10^2 = 100$ it would be correct to guess that $10^1 = 10$. A convincing demonstration of this is the statement $10^1 \times 10^1 = 10^2$. The exponent on the right must equal the sum of the exponents on the left, so it follows that each exponent on the left must have a power of one since $1 + 1 = 2$.

Lastly, we turn to 10^0. Does this mean anything or are we pushing the notation beyond its usefulness? Consider the problem 100/100 or $10^2/10^2$. The answer is 1. Returning to the rule for division, we find the answer is 10^{2-2}, which is 10^0. The implication is that 1 and 10^0 are the same thing. This is true for any number raised to the zero power, **because the zero power signifies a number divided by itself**.

Whenever a number takes the form 10^a, the 10 is called the base. Everything we have discussed is applicable for any base. In other words, $5^3 = 5 \times 5 \times 5$, $4^1 = 4$, $9^0 = 1$, $7^5/7^2 = 7^3$, and so on. Ten was chosen for its simplicity; since multiplication by tens is just a matter of counting zeros.

Negative Exponents

There is still another aspect to dividing powers of ten that must be addressed. What does the problem $10^3/10^5$ mean?

Rewriting the problem as: 1,000/100,000 yields an answer of $\frac{1}{100}$. Following the rule discovered for division with exponents gives $10^{3-5} = 10^{-2}$ for the answer. (Recall that when a larger number is subtracted from a smaller number the answer is negative. For example, $20 - 30 = -10$. You can think of a negative number as representing debt—if you owe \$30 but have only \$20 then you would enter $-\$10$ in your checkbook. Negative numbers are also any numbers to the left of zero on a **number line**—see Figure 7.1 in Chapter Seven page 125 for a description of the number line.) Therefore, 10^{-2} and $\frac{1}{100}$ must equal each other since both are solutions. The same line of reasoning would mean $\frac{100}{1,000,000}$ is 10^{-4}. Is there a recognizable pattern?

$$10^{-2} = 1/100 = 1/10^2$$
$$10^{-4} = 1/10{,}000 = 1/10^4$$

A good guess for a general rule might be: $10^{-a} = 1/10^a$. You might try a few more examples to convince yourself of this and also try to understand that this "new rule" is not new at all, but a result of the previous rule for division. Elegant notational systems often reveal unexpected results.

Negative exponents are another way to express fractions and decimals. Consider the equality $.007 = 7 \times 10^{-3}$. How was this equality determined? Three "different" views are provided below:

1) $.007 = 7 \times .001$, but $.001 = 1/1000 = 1/10^3 = 10^{-3}$ so we have 7×10^{-3}.
2) $.007 = 7/1000 = 7 \times 1/1000 = 7 \times 10^{-3}$.
3) $.007 = 7 \times 10^{-3}$.

Therefore the -3 exponent means move the decimal point three places to the left. **(Any negative exponent shifts the decimal to the left, just as a positive exponent shifts the decimal to the right.)** Since the 7 in 7×10^{-3} has an implied decimal point (7.) three places over from this would give .007 as required.

Three Examples

1. On average, how many twelve-ounce soft drinks were produced in 1990 for each person on the planet? (Stated another way: What was the per capita twelve-ounce soft drink consumption worldwide in 1990?)

In 1990, eighty-five billion twelve-ounce soft drinks were produced, (see page 56) and there were approximately five billion three hundred million people on the planet.

Therefore:

8.5×10^{10} drinks/5.3×10^9 people = $(8.5/5.3) \times (10^{10}/10^9) = 1.6 \times 10^1$ drinks/person.

For values such as 1.6×10^1, scientific notation is often disregarded and the number is simply written as 16, especially if there is little ambiguity concerning significant numbers as in this case (explained on page 85).

2. Livestock production is highly consumptive and polluting. Every second 230,000 pounds of excrement is generated by livestock.[2] How much is this per year?

Since there are 86,400 seconds in one day (60 sec/min × 60 min/hour × 24 hours/day = 86,400 sec/day = 8.64×10^4 sec/day), and there are 31.536×10^6 seconds in one year (3.65×10^2 days/year × 8.64×10^4 sec/day = 31.536×10^6 sec/year). This number must be multiplied by 230,000 pounds per second. Therefore: 2.3×10^5 pounds/sec × 31.536×10^6 sec/year = 72.53×10^{11} pounds/year or 7.253×10^{12} pounds/year (7,253,000,000,000 pounds/year, or approximately 7.25 trillion pounds/year). Note, if we are true to significant figures, then 7.25 should be rounded to 7.3 since the 2.3×10^5 pounds/sec has only two significant figures.

3.a) If the bluefin tuna had been eaten to extinction in 1970, what would each person's share of tuna have been worldwide?
b) Same question but for 1990?

a) On page 56 it was stated that there were two hundred fifty million bluefin tuna in 1970. World population in 1970 was three billion seven hundred twenty-one million people.[3] Let's make the assumption that, on average, each fish has 750 pounds of edible flesh. (Bluefins are huge fish, which can live in excess of thirty years and reach a weight of at least 1500 pounds.)[4]

Therefore: $2.5 \times 10^8 \times 7.5 \times 10^2$ pounds/3.721×10^9 people = $(2.5 \times 7.5/3.721) \times 10^{10}/10^9 = 5.04 \times 10^1$ pounds/person, or simply 50.4

pounds per person. (Using only two significant figures we should state the answer as 50 pounds.)

b) In 1990 there were 20,000 bluefin tuna and world population was five billion three hundred seventeen million.[5] For 1990 we have: $2.0 \times 10^4 \times 7.5 \times 10^2$ pounds/5.317×10^9 people = $(2 \times 7.5/5.317) \times 10^6/10^9 = 2.82 \times 10^{-3}$. Using two significant figures the answer should be 2.8×10^{-3}. Without scientific notation the answer would be written as: .0028 pounds per person or about .045 ounces per person! In other words, whereas in 1970 each person could have had over 2 ounces of fish every day for a year, today there isn't enough remaining for each person to really know what it tastes like. Indeed, instead of the 2 ounce per day figure for 1970, the 1990 figure is roughly one ten-thousandth of an ounce per day. The above results can be calculated by recalling there are 16 ounces to a pound. Therefore, .0028 pounds × 16 ounces/pound equals **.045** ounces, and 50 pounds × 16 ounces equals **800** ounces. Each number divided by 365 gives the numbers quoted above.

* * *

A final note on negative exponents. Just as RBNs are given names, so too with Really Tiny Numbers (RTNs). Using seconds again, we have:

10^{-3} (.001) seconds = 1 millisecond (a thousandth of a second)

10^{-6} (.000001) seconds = 1 microsecond (a millionth of a second)

10^{-9} (.000000001) seconds = 1 nanosecond (a billionth of a second)

10^{-12} (.000000000001) seconds = 1 picosecond (a trillionth of a second)

There is, of course, no limit to RTNs but such numbers lack practical application for most people.

Order of Magnitude

You don't hear many people say, "Last year I earned $25,387.31". More likely, the quoted figure would be $25,000. Rounding off numbers is a useful technique for getting a "feel" for the size of a number, without becoming hampered with exact figures. Sometimes, trying to be too exact can get in the way of creative thinking, where more energy goes into detail than is necessary.

Much of the time we are not interested in exact figures, just a reasonable estimate. (Recall the oil example where we rounded the number of seconds in a day to one hundred thousand.) When the national debt is cited as over four trillion dollars, it is understood the debt is closer to four trillion dollars than five trillion dollars. Ten or twenty million one way or the other is rather insignificant. Remember, a million is only one-millionth of a trillion. Even if we were off by as much as four hundred billion dollars, it would still put us within 10 percent of the correct figure. Often, it is more illuminating and efficient to concentrate on a "power of ten" solution, rather than aiming for an exact answer. The phrase "order of magnitude" is a synonym for power of ten.

The national debt is 12 orders of magnitude greater than one dollar; therefore it's in the trillion dollar range. How many orders of magnitude is a trillion greater than a billion? Three, since a billion is 10^9, and a trillion is 10^{12}. Three orders of magnitude correspond to a factor of a thousand, six orders to a million and nine to a billion. When someone says, "His figures are off by an order of magnitude," they mean he's off by a factor of ten. That's generally significant; it's the difference, for example, between an annual salary of $25,000 versus $250,000.

Enrico Fermi (1901-1954), an Italian-American physicist, was instrumental in developing the atomic bomb during World War II. He also made many peacetime contributions to science and is generally considered one of the most eminent physicists of the twentieth century.

As a teacher, Fermi was known for giving his students problems which initially seemed impossible to solve. Many were of a technical nature. Others, although somewhat trite, were used to illustrate his technique. One famous problem is: How many piano tuners are there in Chicago? Short of going for a phone book, most of us would be at a loss in finding a solution. Fermi, however, was often able to arrive at answers to such questions in a matter of moments without the need of pencil and paper. Such "solutions" are possible because Fermi problems do not demand exact answers. They ask only for reasonable answers, where "reasonable" is defined as having the correct order of magnitude.

Common sense and basic information play a crucial role in all Fermi problems. Before attempting to "solve" the piano tuner problem, consider some "solutions" that are unreasonable.

Would a million (10^6) piano tuners be a reasonable number? No, of course not. How about 10^5 or 10^4? What would you guess for the lowest reasonable order of magnitude? This is certainly no way to arrive at an answer, but it does begin to narrow the possibilities, as well as forcing us to use a common sense approach.

Deciding upon a reasonable population range for Chicago is a necessary first step. A guess of at least one million but no more than ten million is probably sound. If we start with one million, it would then be easy to multiply our final answer by two, three ..., depending on how many millions Chicago actually has.

If Chicago has a million residents, there would be 250,000 families, assuming, on average, four people to a family. Perhaps one family in five has a piano, this amounts to 50,000 pianos (250,000/5). (Implicit in our assumptions are that the number of pianos will supply the demand for some sustainable level of piano tuners.) If one piano tuner can tune, on average, four pianos a day and works 225 days a year (this number allows for weekends, holidays, two weeks vacation, and sick days) then he can tune 900 (4×225) or roughly 1000 pianos a year. If pianos are tuned about once every five years then 10,000 need tuning each year (since there are 50,000).

Therefore, (10,000 pianos per year)/(1,000 tunings per tuner) = 10 piano tuners for each one million Chicago residents. Since Chicago's population is closer to three million, this gives 30 piano tuners. From an order of magnitude point of view, the answer is between 10 and 100 (10^1 and 10^2).

Clearly, many of my assumptions could be challenged. But most answers to this problem, for any set of reasonable numbers and assumptions, will be between ten and one-hundred. This is what makes the process valuable. Though the details may never be known with certainty, a plausible and convincing range of values can be determined.

* * *

Fractions, percentages, scientific notation, and order of magnitude calculations are all useful in explaining and calculating a wide range of modern day problems. They should not be viewed as isolated concepts, but as a set of elements that can be brought together to deepen understanding. Like musical notes, they each have their own identity, but when combined, they are capable of much greater expression. In the chapter that follows, three problems are examined in great detail which use many of the mathematical concepts we have previously discussed.