CHAPTER THREE

A Brief History of Mathematics

Precounting and Primitive Counting

Historically, what we have needed to know depended on where and when we lived, and who we were. Humans as hunters and gatherers needed few number skills. A sense of "larger and smaller than," an ability to pair-off two different sets of objects, and some modest counting skills, would have made a person very mathematically literate in paleolithic times (40 thousand years ago). Alternatively, the early Stone Age dweller who could not evaluate which pile of food was larger probably stood a dim chance of passing on any progeny. From a paleolithic perspective then, simple counting skills would have exhausted mathematical knowledge.

In precounting societies, an ability to form a one-to-one correspondence between different sets of objects would be immensely helpful. For example, counting is unnecessary in pairing one male to one female (assuming a monogamous society). Those left unpaired, being the same sex, might be encouraged to seek greener pastures elsewhere or to play a differentiated role within their group. Anthropological correctness aside, this pairing ability would be invaluable in a primitive communal setting—even if the task is just to distribute an equal share of berries to each member of the tribe.

No one knows exactly when the ability to pair off objects emerged for man. But evidence of bones with notches carved into them suggests that humankind has been keeping records in this manner for at least thirty or forty thousand years.[1] Many scholars believe that this mathematical innovation led eventually to written language.[2] The same idea is still practiced today, when a line is drawn each time an event occurs. It's the old cliche of the prisoner who scratches a mark on the wall each day to keep track of time. A notable version of this concept is the story of Cinderella. Remember how the prince goes from lady to lady trying to find the best fit for the glass slipper? Here there is only one acceptable pairing, yet numerous unsuccessful correspondences are required in order to finally isolate the one correct match. Cinderella belongs to the genre of stories where the hero must find a unique correspondence (for example, the matching pieces of a cryptic medallion) that will prove he is the rightful heir to the throne, or give him the power to claim any throne he wants.

The practice today of receiving a receipt for a business transaction is a sophisticated example of the above idea. When paleolithic man bartered he probably employed a similar method of keeping track of the number of items in question. Both parties could keep a bone with the same number of notches representing the trade. The same idea exists in the form of the modern tally stick, which governments, such as Great Britain, used into the early nineteenth century for tax keeping purposes.[3]

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During the last several centuries Western man has discovered populations of indigenous people who live as their stone age ancestors did. Studying their cultures has helped to advance our understanding of the way number systems evolved.

Counting systems were no doubt long in the making. After all, the sophistication of a counting system is linked to the demands of the people who use it. The amount of numbers required in a number system is directly tied to the complexity of the society in question. It would be difficult, for example, to imagine a paleolithic people using even three digit numbers. Not because of a lack of intelligence in dealing with such numbers as one or two hundred, but because of for any practical application of them.

George Gamow, in his classic book, *One, Two, Three … Infinity*, tells of an African tribe called the Hottentot who do not have names for numbers beyond three. The Hottentot count: one, two, three, many.[4]

In Graham Flegg's book, *Numbers: Their History and Meaning*, there is the example of another African tribe called the Damara of Namibia who cannot count beyond two. To quote Flegg: "… the Damara of Namibia, who were prepared to exchange more than once a sheep for two rolls of tobacco but would not simultaneously exchange two sheep for four rolls."[5]

All counting systems display patterns, but the more primitive the system, the faster the onset of repetition. The following examples from Flegg will clarify this idea. A tribe of aborigines living near the Torres Strait between Australia and New Guinea count only up to five as reproduced below:

one: urapon
two: okosa
three: okosa-urapon
four: okosa-okosa
five: okosa-okosa-urapon

Another aboriginal counting system goes up to six as follows:

one:	mal	four:	bularr-bularr
two:	bularr	five:	bularr-guliba
three:	guliba	six:	guliba-guliba

The above two number systems have the patterns:

one	two-two-one	one	two-three
two		two	three-three
two-one		three	
two-two		two-two	

Counting, though seen as a simple procedure, is not. It probably took thousands of years for human beings to understand the concept of a number devoid of an object—an understanding of "twoness" as opposed to two "somethings." The various ways the number two is expressed today in English may serve as a linguistic fossil record of our own slow progress toward number abstraction.

Consider some of the synonyms for the number two: pair, couple, twain, brace, yoke, deuce, couplet, set, team, and twin. On the other hand, we do not have a multitude of synonyms for numbers like 253. Larger numbers probably came into use after we had abstracted counting; when we no longer needed the concrete crutch of associating a number with a particular set of things. It is revealing to find a word like thrice in our own language with the double meaning of three and greatly. A throw-back to when we counted one, two, many? In fact, in the Indo-European counting systems, the "early words for three can be linked to a root word whose meaning was beyond."[6]

Everyone realizes there is no limit to counting. Our present day number system can accommodate any number large or small. But how well do people use and understand large and small (microscopically small) numbers today? Anyone who doesn't know there are two tens in twenty is handicapped in our society. What about the person who doesn't know how many millions there are in a billion, or a trillion? It is plausible that many educated people today count: 1,2,3...1,000,000, many. Is it possible that these individuals are as limited within our own society as an aborigine is within his own culture, who can't count beyond two? How then, has the need for numeracy changed throughout history, and how much do people really need to know today?

Number Systems

There is a formidable gap in our understanding of the evolution of mathematics. At best, we can make some educated guesses about pre-counting societies and primitive counting systems. But thousands of years stand between these skills and a developed number system employing arithmetical principles. By the fourth to fifth millennium B.C. there is clear evidence that mathematics of a highly abstract nature already existed. The oldest known mathematical document

was written by an Egyptian named Ahmes around 1700 B.C.. It is commonly referred to as the Ahmes or Rhind Papyrus. (Rhind was a Scottish antiquarian who purchased the papyrus in 1858.) We know from Ahmes that the papyrus he wrote was based on even earlier works; possibly from as far back as five thousand years ago. Ahmes refers to his work as "… the knowledge of all existing things and all obscure secrets."[7] We see that mathematics was not thought of as belonging to the masses, since "obscure secrets" were usually reserved for the priesthood.

Ahmes had 84 problems on the papyrus.[8] Many of these problems were algebraic. Evidently, solving for an unknown quantity is a very old concept. And since Egyptian mathematics centered around practical problems, algebra must have been used as long as five thousand years ago with problems involving weighing, measuring, and land surveying and parceling. None of the problems were explained and only rote instructions were given for solutions. Thousands of years later in Europe during the Renaissance, mathematicians were still reluctant to divulge their proofs. One reason for this was that problems were often given as challenges and as a means of promotion or instatement in universities. It appears this tradition of vagueness is alive and well in many schools today. All too often, mathematics is taught as a series of steps that magically produce correct answers.

Because of the Ahmes Papyrus, scholars believed for many years that Egyptian mathematics was the oldest and most advanced. However, about one hundred years ago, archaeologists learned to read the Summerian wedge shaped writing, Cuneiform, which is the oldest writing known. (If not for the discovery of the Rosetta Stone in 1799, Egyptian Hieroglyphics would have remained as much a mystery as Summerian Cuneiform.) A knowledge of Cuneiform led scholars to conclude that the Babylonians were more adept at mathematics than the Egyptians. It is now believed that the Babylonians developed the first place-value system in the world.

The place-value system is taught in the early elementary grades; remember learning about the ones, tens, hundreds, thousands, …

columns? Well it took thousands of years before some clever Babylonian thought up the novel idea of using position to describe amount. Here was a way to use a small number of symbols to represent any quantity. Take, for example, the numerals 1 and 9. Ordering the numerals as 1 followed by 9 gives nineteen; in reverse, it is ninety-one. In our present system we use 10 symbols (0-9) and depending upon how we place them we can write any number. Contrast this to Egyptian numerals, where position played no role. The number 12 would have been understood whether written: ∩|| or ||∩

(*See Figure 3.1 below for a list of Egyptian numerals*).

Egyptian Hieroglyphic Numerals

Figure 3.1

The Roman number system is a bit more sophisticated than the Egyptian, though not as useful as the earlier Babylonian system. With Roman numbers, proper positioning serves as an abbreviation or short-cut by using a subtraction principle. For example: IV = (5-1) instead of IIII, IX = 10 - 1 instead of VIIII and in general, whenever a larger number is preceded by a smaller number it is understood to mean larger minus smaller, as in other cases like, XL = 50 - 10 and CM = 1000 - 100. Examples of several other number systems are given in Figure 3.2 below.

The number 23 expressed in ancient numerals.

. . | | | (Cretan)
Δ Δ | | | (Greek Attic)
× × | | | (Roman)
✧ ✧ | | | (Aztec)

Figure 3.2

A lack of concise notation, a place-value system, and an under-

standing of the zero concept, impeded the mathematical progress of ancient cultures. Imagine multiplying or dividing with Roman numbers. Though there are ways to do so, they are long and involved.[9] Simple calculations today required the talents of expert “mathematicians” during antiquity.

The most used number system in the world today is the decimal system which is based on the number ten, as opposed to the duodecimal system that has twelve as a base. The Babylonians, however, used a system based on 60 rather than 10, called the sexagesimal system. Therefore, instead of having a one’s column ranging from 0-9 theirs was 0-59. The second column was the 60’s, the third column, 60×60, (3600’s column), the fourth column $60 \times 60 \times 60$, and so on.

The Babylonian sexagesimal system is still in use today. Circles are divided into 360 degrees (6×60), and the hour and minute are multiples of sixty. Sixty is a very convenient number; it can easily be divided by: 2,3,4,5,6,10,12,15,20, and 30. Ten has only 2 and 5 as divisors, thus making 60 a simpler attractive base for purposes of division.

A little imagination helps to see how decimal systems and finally place-value systems originated. Before the development of fully operational counting systems, people may have begun to finger count to ten by pairing one object to each finger.

For numbers larger than ten, transactions could be carried out by counting sets of ten. One can imagine a farmer trading 7 sets of ten bushels of wheat for 5 sets of ten bushels of barley, or a shepherd describing the size of his flock as 3 sets of ten. We have remnants of this type of counting in our language today: The “score,” for example, which represents 20, is most remembered in Lincoln’s Gettysburg Address that begins, “Four score and seven years ago … ”, i.e., 87 years ago.

It was in fact quite common to keep track of the numbers of tens, hundreds, thousands, and greater multiples of ten, by placing a pebble aside for each set of ten, hundred, thousand, etc.. You can begin to see how place-value systems could develop from this—a column for counting to ten, a column for pebbles that represented the number of tens, a different set of pebbles (larger ones) after ten sets of ten were reached and so on. In fact, one of the oldest calculating machines, the abacus, is based on the principle of separating numbers into ones, tens, hundreds, etc..

Our present number system is very abstract. Yet most people see it

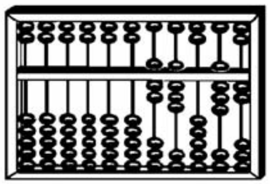

Figure 3.3 - An Abacus

as the concrete part of mathematics education today. “I can deal with numbers, but when he starts putting x’s and y’s on the board, forget it!,” is a common retort of many students. But the only reason people deal successfully with the number system is because they are psychologically acclimated to the symbols and their meanings. Most people do not actually understand number systems. If they did, they would be able to tell you that in base 2, the number 1110 is the same as 14 in base 10. Nor would there be any opposition to converting from the present duodecimal system, used for weights and measures, to the metric system—which is decimal! (See Appendix A. page **179** for a discussion of different bases.)

In general, numbers are more difficult for us to deal with than pictures. Euclid formulated the axiomatic basis for plane geometry (picture based) in 300 B.C., while the proper group of axioms for our algebraic system (number based) was not deduced until this century.

Zeroing in on Zero

Another leap made by the Babylonians was in recognizing the need for zero. It wasn’t until about the third century B.C. that they introduced the symbol (⋜) for zero.[10] Before this, the best they managed was a blank space, which is a precarious representation of zero. It makes for notational headaches and grand scale ambiguity. How can numbers like 64,000 or 6,040 or 604 be written with clarity? The Babylonians resorted to writing 604 as 6 4 (with their own numerals of course). There is little room for sloppiness here. Even after the Babylonians devised a symbol for zero, they used it only between numbers,[11] never at the end of numbers and never in calculations.[12] It was as if they were willing to concede that zero was a necessary component for a place-value system, but they still wanted as little to do with the concept as possible.

Even the Greeks who started us on the road to mathematical rigor, never dealt with it successfully. It wasn’t until the Hindus developed their version of the zero concept, somewhere between 1000 and 1300 years ago, that the modern usage and symbol for zero were adopted. The intriguing question is why ancient cultures had such difficulty with the concept of “nothingness.” In all likelihood, it was their inability to reconcile the notion of “nothingness” as something concrete. How, for example, could one justify the value of 10, by using the number 1 and a symbol with no value, 0?

Another more mysterious problem arises with division and zero. Division by zero is said to be undefined, yet division into zero (by 1 for example) results in zero. If zero was to be considered as merely another number, why the restriction concerning division? This must have troubled the ancients almost as much as it does present day students, when they are first confronted with the rule “thou shalt not divide by zero.”

A little thought reveals why dividing by zero causes problems. There are an infinite number of points between the numbers zero and 1. The number ½ is twice as close to zero as 1 is. The number $\frac{1}{10}$ is ten times closer to zero than 1. The closer we get to zero, the smaller the numbers become. Instead of dividing 1 by zero, what values would result if we divided 1 by numbers that were a hair’s breadth away from zero? One divided by ½ is 2, one divided by $\frac{1}{10}$ is

10, one divided by one-millionth is one million. The closer we approach zero as a divisor, the larger our answer becomes. The pattern is clear, as our divisor approaches zero, the answer continues to become larger. In the ultimate case of division by zero, we would have to say the answer is infinite and therefore undefined. (It should be understood that the above argument holds when smaller and smaller quantities are divided into numbers other than zero or infinity).

A similar argument shows that 1 divided into zero equals zero. If we divide 1 into 1, the answer is 1, but if we divide 1 into ½, one-half results. Continuing on (one divided into one-millionth, is one-millionth) we see as the dividend approaches zero so does the answer.

Confusion with Fractions—An Ancient Dilemma

Another difficult subject for the ancients was fractions. In fact, they avoided them as much as possible.[13] Fractions are strange; they have a way of changing their context without the mind realizing what has happened.

An employer with ten workers may have a payroll of five thousand dollars a week. A worker who receives five hundred dollars for the week does not regard his earnings as a fraction of five thousand dollars, but as a whole unit. A piece of pie is a fraction of a pie, but to the eater it's a **whole** piece. A person might eat a large piece of pie followed by a small piece. Two pieces were eaten, but how much of the pie was consumed? Understanding how to divide a thing into an odd number of pieces or adding two different size pieces is conceptually difficult.

Adding or subtracting fractions with different denominators means working with different pieces. The only way to work with them is to employ the concept of conversion, just as in adding feet and inches. Finding the "least common denominator" is a conversion process with each piece redefined into smaller units. In order to add one-third and one-fifth, the proper "conversion" is to cut the one-

third into five equal pieces and the one-fifth into three equal pieces. Doing so, one finds that all the little pieces are now the same size and it would take fifteen of them to make a whole. There are, however, only eight pieces available so the total is eight-fifteenths. This process is the visual counterpart of finding the least common denominator (which for this case is fifteen).

Without a well-developed mathematical notational system, the kind of fraction arithmetic discussed above becomes very complicated. As previously stated, the ancients did all they could to avoid fractions, including creating finer (smaller) units of measurements so that a fraction could be restated as a new "whole" smaller unit.

The Egyptians did all their fraction arithmetic by re-expressing fractions as sums of unit fractions (fractions that have one in the numerator). For example, $5/7$ was re-expressed as, $1/2 + 1/7 + 1/14$, before attempting any work with the number. There was little conformity among ancient people in working with fractions. Cajori tells us: "In manipulating fractions the Babylonians kept the denominators (sixty) constant. The Romans likewise kept them constant, but equal to 12. The Egyptians and Greeks, on the other hand, kept the numerators constant, and dealt with variable denominators."[14]

Regardless of the difficulties the ancients had in working with fractions, they needed them. Therefore, societies at least four millennia ago used many of the skills taught today in seventh-grade. Though most people during this time were illiterate, the upper crust of society—the priests, engineers, scribes, bureaucrats, and skilled technician—all required mathematical knowledge. Just as today, the level of mathematics training in ancient societies depended on the individual. The level of universal literacy and numeracy that exists today (or is attempted) does so out of need, not altruism.

Money and Numeracy

Trade existed for thousands of years without the need for money. As far back as 5000 years ago, the Chinese emperor Shen Nung, set up what was probably the first "retail" market in the world.[15] No money exchanged hands, all transactions were bartered.

The natives of the Pacific island of Uap, well into the twentieth century, used "money" called fei, made of huge stone disks with holes bored out of their centers. Some of this "money" was so heavy it took at least two grown men to carry one "coin." Almost anything you can think of has been used for "money": sea shells (wampum), tobacco, animals, slaves, coconuts, tools, grains, salt, and silk were all used as a means of exchange.

Ancient societies before Greece were moneyless societies. Even though large scale trading existed between ancient countries, not a coin was ever exchanged. The Egyptians traded with the Phœnicians for wood by the boat load, and had generations of lumberjacks hacking away at the cedars of Lebanon.[16] But the wood never went through the hands of a middle man or out onto the open market; it was for the temples the Pharaohs built to honor their gods and themselves and to maintain the royal navy. Everything was run by the state. Each person had his or her particular task to perform for which they were assured a place to sleep and food to eat. The idea of "getting ahead" in the ancient world (with one exception), or in later years feudal Europe, did not exist.

Mesopotamia (ancient Babylon) may have offered the only promise for a better lifestyle in the ancient world, as described below by Charles Van Doren.

> Perhaps no other civilization besides our own has been so dependent on literacy, even though probably only one percent or fewer of Mesopotamians were ever literate, even in the best of times. Scribes, who wrote letters and kept records and accounts for kings and commoners alike, always possessed great power. As ancient advertisements for pupils and apprentices proclaimed, scribes wrote while the rest of the people worked[17]

Moneyless lifestyles have been common throughout history. The English manor, the French seigneurie, and the German Gutsherrschaft, are all examples of self sufficient entities that existed without benefit of money.[18] The colonial period in America and frontier life were in many respects similar to such institutions.

Money, as we know it, was invented by the Greeks around the eighth century B.C.. This surprises people because it is difficult to picture a system of taxation, which the Egyptians and other ancient civilizations used, without money. But taxes were paid with grain or by a certain weight of silver or gold, but not currency as it is defined today. Buying, selling, and taxation in the contemporary sense did not exist.

The Bible can be a source of confusion concerning money concepts. In the seventeenth chapter of Genesis, God commands Abraham to circumcise any male who is eight days old, whether born in his house or "bought with money of any stranger." But according to Norman Angell in his book, *The Story of Money*, "The word here translated 'money' is in the original keseph; in the Septuagint it is correctly rendered by apyupiov, and in the Vulgate argentum; in fact, it should have been translated 'silver,' not 'money.'"[19]

When the Greeks introduced coinage in the late eighth century B.C., they laid the foundation for our modern world. Money is freedom. Without money, a fair exchange is difficult and time consuming. Only in static societies where commodities never change, in kind or quantity, can simple barter systems work efficiently. Otherwise, you may be trading a cow for 100 pair of shoes just to get your cow's worth. With money came opportunities unthought of in moneyless societies. Money made societies more dynamic and complex. The necessity to understand a denominational system—computing costs, making change, and paying taxes—required more computing skills for a larger part of humanity. All free men were now obliged to learn money concepts in order to function in their world. It became more important for people to use fractions in daily affairs. Being able to make change and understand new taxes became part of the new literacy and numeracy. Even those who looted and killed needed to become money literate.

During the sixth century B.C., Athens experienced the first money crisis in history. Too much easy credit caused too many commercial ventures which failed (sound familiar?). This put the borrowers in debt and left the lenders empty-handed. The importation of cheap grain from places like Italy by Greek merchants caused Athenian grain to be less valued, which in turn only caused more people to go belly-up. The situation was worsened when small farmers used their wives, children, and themselves for collateral (human chattel). Lenders, who took possession, had to feed and cloth their property, but since Athenian agriculture was in a slump, and slaves were abundant, the lenders were better off not collecting. This was a monetary disaster that threatened the fabric of Athenian culture.

Though the Greeks suffered other money crises, they did manage to patch things up for a while. Solon, a popular Athenian, is credited with implementing an innovative and successful economic plan. As part of his plan he outlawed human chattel and devalued the currency by 27 percent.[20]

Any time money is devalued it hurts those who lend, since the money they are paid back with is worth less than the money they loaned. As an exaggerated example of devaluation, let's say each dollar is devalued so that it is worth only ten cents. This means each dollar has lost 90 percent of its value or has been devalued 90 percent. Since the value of the merchandise hasn't changed, more money will be needed to purchase the same goods. Since each "dollar" is now worth only ten cents, ten "dollars" is needed to purchase what before cost one dollar.

This was the only instance in Athenian history that such an intentional devaluation occurred.[21] Solon's economic reform package involved much more than outlawing chattel and currency debasement, but we needn't go further into it for our purposes. Suffice it to say, though many were displeased, his reforms were successful. And if you're putting yourself in the position of the lender, remember, Solon only devalued the currency by 27 percent. Which means 100 pennies would have the buying power of 73 pennies ($100 - 27 = 73$) or

roughly, it would take about four dollars to buy what had cost three dollars previously. This can be seen by rounding 27 percent to 25 percent for ease of calculation. With each dollar worth 25 percent less (or 75 cents), 4 new "dollars" equals three old dollars. Clearly, this type of monetary manipulation causes a lack of faith in the currency of a nation. It also goes to show why printing too much money causes inflation; the more dollars in circulation the less they are worth and so your money buys less and less. From a practical point of view, inflation and devaluation are the same thing. (Inflation will be discussed more in chapter six.)

The Golden Age of Greece and Mathematical Thought

The period 600-300 B.C. is considered the Golden Age of Greece. Not surprisingly, Greek mathematics began to flourish at this time. Pythagoras (who may never have existed) is credited with forming a group called the Pythagoreans who were possibly the first to do mathematical research.[22] Most of Greek mathematics and science was based on geometry and arithmetic with whole numbers. (Fractions were used, but the Greeks considered them only as ratios of whole numbers.) Greek mathematics during the early part of the Golden Age was inseparable from metaphysics. When it was discovered that certain numbers were not expressible as simple fractions, a mathematical, as well as a metaphysical, crisis arose.

Consider a right triangle whose legs (base and height) measure one inch. (Figure 7.6 in Chapter 7 page 135 can be considered a generic right triangle.) The hypotenuse (the line connecting the two legs) will have a length of the square root of two inches. The square root of two cannot be represented as a ratio of whole numbers and is therefore defined as an **irrational** number. Furthermore, any attempt to find the exact value of the square root of two yields an unending, nonrepetitive sequence of numbers. Using a calculator to find the square root of two, fills the screen with numbers beginning 1.41—only the display area of the calculator prevents the numbers from continuing on forever. This was a metaphysical nightmare for people who believed geometry and whole numbers were expressions of perfection. A finite length, easily seen and easily constructed, represented by an unending number? A length not expressible as a ratio of two whole numbers? Such concepts violated the basic premise upon which Greek intellectual and metaphysical thought were based.

The ancient Greeks believed the universe was harmoniously governed by "perfect" geometric structures (circles and spheres, for example) and simple arithmetical ratios. Heaven was perfection, (for Plato), and the Earth was a coarse, shadowy, representation of that perfection.

It could be argued that we still have a preferential way of viewing reality. In principle, we agree with the ancient Greeks regarding perfection and symmetry in nature, though we no longer differentiate between heaven and Earth. The notion that the universe is explainable in terms of the language of mathematics, is at the heart of all physical science today.

Though not bound to the few "perfect" geometric figures of the Pythagoreans, the search today is for mathematical symmetry. Modern physics relies on the "symmetry" of physical laws. Physicists speak of "beautiful" theories, where beauty often refers to mathematical equations that easily describe these symmetries. The brilliant physicist, Paul Dirac, once said he preferred beauty to accuracy in describing reality. One wonders whether we have been subliminally conditioned to search for beauty in nature because of Greek tradition, or if it is an innate part of the human psyche to do so.

By 300 B.C., Euclid (the man responsible for your studying tenth grade geometry) had organized the subject into 10 axioms from which he deduced all geometrical "truths." The organization and methodology that Euclid brought to human thought was analogously as unifying, and evolutionary, as any political reconfiguration the world has ever known.

Though Greece passed on, its accomplishments remained. It enriched the empire of Alexander the Great, maintained Byzantium

while Western Europe fell into darkness, and found a home (via the Hindus) in the Muslim world. It provided for Europe's Renaissance and even today continues to be a source of insight for classical and modern thinkers. Even Rome, an intellectual lightweight at best, saw the value in learning from the Greeks—when she wasn't destroying them.

Numeracy, the Dark Ages, and the Crusades

By 1 A.D., Rome had absorbed Greece and had incorporated many of its economic ideas into her own system. She boasted a middle and merchant class and money was a common financial device. Of course, most Roman citizens were still innumerate. But those who lived in Rome proper found books easy to come by and were fluent in the arithmetic of money.[23]

Gold coins were used for imperial payments, and taxes were payable in gold, silver, and copper coinage; ordinary transactions of every day life were carried out in small denominational copper coins.[24] The same basic money skills we find necessary today were equally important 2000 years ago.

The responsibilities of the educated classes in democratic Greece and republican Rome were similar in many respects to our own. Electing politicians who understood the role of trade and its affect on currency, the need for stability in the market place, and an internally consistent point of view, were just as important then as now. However, when the Western Roman Empire fell in the fifth century A.D., literacy took a nose dive in Roman-controlled Germany, France, England, the Netherlands, Belgium, Switzerland, and Italy. (It is convenient to cite the names of these countries even though none of them existed in their present form at this time.) Western Europe plunged into the Dark Ages, where no middle or merchant class would exist again for over six hundred years. The need therefore for basic arithmetical skills, dwindled with the loss of commerce, coins, and culture. England in 1000 A.D. had fewer educated people than

Rome in 1 A.D.. Western Europe was so backward in 1000 A.D. that even many monks were illiterate. Ironically, Bibles and Greek classics were at times copied by monks who mimicked the script without the slightest idea of what they were writing.[25] (Almost the kind of "work" done in many mathematics classes today.)

Dirk J. Struik puts the level of Western European mathematics during the Middle Ages in perspective, when he writes: "During the early centuries of Western feudalism we find little appreciation of mathematics even in the monasteries. In the again primitive agricultural society of this period the factors stimulating mathematics, even a directly practical kind, were nearly nonexistent ..."[26]

Though mathematics stagnated for hundreds of years in Western Europe, both the Byzantine (Eastern Roman Empire) and the Muslim Empires used and advanced the subject. The Byzantine Empire which extended as far west as Sicily was a catalyst for the revival of trade, commerce, and mathematics in what was once the southern part of the Western Roman Empire. As a result of this, the Italian cities of Genoa, Venice, and Pisa were trading with the Far East by the eleventh and twelfth centuries. In fact, trade, commerce, and competition had accelerated so much during this period, that by the early thirteenth century, Venetian merchants convinced the leaders of the Fourth Crusade to attack Constantinople (the capital of the Byzantine Empire). With Constantinople in Western hands, the Venetian merchants had a more exclusive trading arrangement with the Far East. Though the intent of the Crusades was to liberate Jerusalem from the Seljuk Turks, they played the more important role of exposing backward Europeans to culture and education. They also helped to solidify the West's interest in the Far East, which as we will see later, indirectly led to a renewed interest in navigation and mathematics.

After the Crusades began their revitalizing process, it became imperative for the West to maintain accessible and safe trade routes to the East. But when Constantinople fell to the Ottoman Turks in 1453, Europe lost her route to the East. Many historians place the beginning of the modern age at this date since it initiated a more extensive search for alternative routes east which led to the Age of Discovery.

* * *

There is no simple way to encapsulate the last five hundred years of human history. The changes which have occurred from the late fifteenth century to the present are monumental. If we take a long view of human history, the last 15,000 years for example, we would see humanity modestly moving along an essentially straight-line path except for three events that have significantly altered its course and speed: the Agricultural Revolution, the Age of Discovery—punctuated by the permanent migrations of Europeans across the Atlantic, and the Industrial Revolution.

Our understanding of the events that led up to the Agricultural Revolution is at best speculative. The Age of Discovery and the Industrial Revolution, in contrast, have been etched quite well onto paper and into our minds, thanks to the inventions of writing and complex language.

Modern history is full of examples highlighting the increasing need for numeracy within society. We will therefore conclude chapter three with some historical examples and their relationship to numeracy.

Numeracy and the New World

Fourteen hundred ninety-two is a popular date because it ushered in the age of European exploration and exploitation. As every school child is taught, Columbus believed he had reached the Indies, hence we have "Indians" in North America. Believing he had reached the Indies, however, is at the bottom of Columbus's list of errors. Before going any further, let us dispel the myth that Columbus proved the world was round. This was the furthest thing from his mind. Ancient Greek knowledge that the world was round had reached the West, so any educated European knew this. The real question during the fifteenth century was not the shape of the world, but its size and geography.

Columbus appears to have favored arguments that supported a smaller circumference for the earth and a larger land mass for Asia. This made travel to the East using a western sea route seem more feasible.

Columbus spent ten years trying to convince European monarchs that by sailing west, he could reach the East Indies. Most monarchs, however, had advisors who discouraged the enterprise. Few advisors accepted Columbus's notions regarding the size of the Earth and the extent of Asia.

The bottom line is that Columbus was lucky. To quote Timothy Ferris:

> Columbus's plan appeared foolhardy to anyone who possessed a realistic sense of the dimensions of the earth. To sail westward to Asia, as the geographers of the court at Castile took pains to inform Columbus, would require a voyage lasting approximately three years, by which time he and his men would surely be dead from starvation or scurvy.[27]

Much of what Columbus believed was based more on personal bias than existing knowledge. For example, he put much stock in a reference from the Apocryphal Book of the Old Testament, Esdras(II Esd. 6:42) which says, "Six parts hast thou dried up."[28] From this he inferred sixth-sevenths of the world was dry land.

Believing that he had reached the Indies, Columbus said, "Neither reason nor mathematics nor maps were any use to me."[29] Ironically, he was correct, since no amount of mathematics could predict, and no known map contained, the lands between Europe and Asia. Additionally, the only reason Columbus obtained the necessary funds to set sail was because Ferdinand and Isabella decided they had little to lose and much to gain if Columbus was right; the king and queen played a long shot that paid off.

Some eighteen hundred years before Columbus set sail, the Greek thinker Eratosthenes measured the circumference of the Earth and came to within 4 percent of the modern value. Mathematically, only the concept of **direct proportionality** (taught in seventh-grade) is needed to perform this calculation. We will discuss Eratosthenes's method below and in Chapter 7 extend these ideas beyond this "special linear case."

Before we begin the mathematics there are two things we need to state. These are:

1. The Earth is assumed to be a perfect sphere (it is actually an oblate spheroid).
2. A circle has 360°.

If you travel from the north pole to the equator this represents ¼ of the circumference of the planet. In angular measurement this amounts to 90° (¼ of 360°). It also represents roughly 6000 miles. Since our goal is to find the entire distance around the planet, we can set up a proportion as follows:

6000 miles is to 90° as ? miles is to 360°

or in a simpler form, where all degree measurements are placed on one side and miles on the other:

$$360°/90° = ?\ \text{miles}/6000\ \text{miles}$$

Since the left-hand side equals 4, we must ask, "What number will 6000 go into four times?" The answer is, of course, 24,000. Therefore, the Earth is roughly 24,000 miles in circumference (actually it is closer to 24,900 miles).

If you felt this problem had more steps in it than necessary, you'd be correct! Once you knew one-fourth of the circumference was 6000 miles, all you needed to do was quickly multiply by 4. Eratosthenes, however, did not travel 6000 miles; in fact, he probably did not travel at all. As it turns out, he was told that on the first day of summer in Syene (now Aswan, Egypt) the sun was directly overhead (at its zenith) and cast no shadow. He lived 500 miles due north in Alexandria, and on the same day at the same time (12:00 noon) a stick in the ground at Alexandria casts a shadow of 7.5°. In other words, the angle that Eratosthenes had was not the 90° in our simplified example, but 7.5°. Using the same process as before we have:

7.5° is to 500 miles as ? miles is to 360°

or again, more simply:

360°/7.5° = ? miles/500 miles

If it was as obvious to us that 7.5° represented 1⁄48 of a circle, as it was when 90° represented ¼ of a circle, we again could forego the process of taking a ratio and using degrees, and immediately multiply 500 by 48.

It is uncertain whether Columbus was aware of Eratosthenes result, or for that matter, if he would have cared. From what we know Columbus relied heavily on the writings of the Frenchman Pierre d'Ailly,[30] who was acquainted with the work of Ptolemy—the ancient authority on astronomical and geographical matters. As it turns out, d'Ailly disagreed with Ptolemy about the extent of Asia. d'Ailly believed it was larger than Ptolemy did, thus shortening the voyage westward. On the other hand, Ptolemy disagreed with Eratosthenes about the size of the Earth, claiming it to be smaller. Columbus, in his effort to make the voyage as easy as possible, followed d'Ailly concerning the extent of Asia, and followed Ptolemy concerning the circumference of the Earth. Columbus's reasons were hardly scientific. Basically, they were a mixture of wishful thinking and religious dogma. Ironically, it was Eratosthenes's approach that guided Ptolemy and the geographers of the Renaissance, including Columbus. The problem was one of miscalculating the initial distance between two north-south points, which led to an incorrect proportion between surface distance and angular distance. Ptolemy and several prominent Renaissance geographers were led to calculate a

smaller globe due to erroneous information. Columbus's error was in favoring such calculations.

It is only fair, however, to give Columbus his due. Though fortune was with him by obstructing his voyage with two large continents, travel in an east-west direction during this time period was perilous. A major difficulty for navigators up until the second half of the eighteenth century was in determining location along an east-west (longitude) line. North-south measurements (latitude) were easily taken by observing the change in location of a given star, or the sun, with respect to the horizon; such methods to determine latitude had been used for thousands of years. But discerning longitude is a nontrivial problem requiring a complicated mathematical process, unless a precise time piece is available. Since no seaworthy clock of this precision existed until 1761, navigators had to be mathematically able. A quote from Daniel Boorstin highlights the growing role of mathematics during this period:

> … the problem of longitude (was) an educational as well as technological problem. The great seafaring nations optimistically organized mathematics courses for **common sailors** (my emphasis). When Charles II set up a mathematics course for forty pupils at Christ's Hospital, the famous 'Bluecoat' charity school in London, teachers found it hard to satisfy both the sailors and the mathematicians. The governors of the school, noting that Drake, Hawkins, and other great sailors had done well enough without mathematics, asked whether future sailors really needed it. On the side of mathematics, Sir Isaac Newton argued that the old rule of thumb was no longer good enough. 'The Mathematicall children, being the flower of the Hospitall, are capable of much better learning, and when well instructed and bound out to skilful Masters may in time furnish the Nation with a more skilful sort of Sailors, builders of Ships, Architects, Engineers, and Mathematicall Artists of all sorts, both by Sea and Land, than France can at present boast of.' Samuel Pepys, then Secretary to the Admiralty, had already set up a naval lieutenant's examination which included navigation and, following Newton's advice, naval schoolmasters were actually put on board ships to instruct the crew in mathematics.[31]

I have gone to the trouble to extensively quote Boorstin because we see a similar attitude, voiced by the "governors of the school," regarding the value of mathematics that many people still hold today—namely, "What do we need it for?" Newton's quote also gives us a sense of how relevant mathematics was becoming to the average British citizen from a political vantage point, as well as the upward mobility that an education in applied mathematics could afford a "common" person.

There is an interesting parallel here between the literacy we associate with the written word and mathematics. No scribe in ancient times worried about going hungry. The ability to read and write assured an individual a comfortable lifestyle, free of manual labor; few people had such fortune. But something very interesting has occurred since the Industrial Revolution of the mid-eighteenth century. Reading and writing became necessities for everyone. During the early years of the twentieth century, universal literacy permitted a person to stay monetarily afloat (usually), avoiding many turbulent economic storms. However, since 1950, the basic skills of universal literacy no longer assure one of a better life. What has become more apparent during the second half of the twentieth century is the ever increasing value of a good math and science background. After all, the person working on an assembly line in 1950 was probably as well educated, if not more so, than those ancient scribes who lived better than most people 3000 to 4000 years ago. But today, the equivalent of an ancient scribe is one who is well versed in mathematics and science, in that such skills set him apart from the rest of society and hold so many additional life options and opportunities.

Seamen back in the eighteenth century put algebra, geometry, and trigonometry to use in the practical world of navigation. Still, it would have been preferable to have an efficient time keeping device to eliminate complicated mathematical calculations. As we all know, the more intricate a mathematical process (or any process) the more room there is for error. A slight miscalculation in longitude could mean anything from a delayed ship to one that ran aground. Delays could have serious economic and military consequences wherein

companies could lose large profits and naval commanders might lose the battle, if not the war.

The importance of longitude cannot be overstated. Its relationship to practical mathematics and astronomy, and its historical connection to commerce and war are formidable. A brief description of it follows.

Since the Earth rotates west to east, (looking down from the north pole, the Earth would appear to turn counter-clockwise) it is possible to determine your position if you know the time difference between your present location and some reference point. Since it takes the sun 24 hours to apparently complete one revolution around the Earth (meaning it appears at the same point in the sky in 24 hours) the sun must be traversing 15° each hour: 360°/24 hours = 15°/hour. Therefore, we can relate one hour to fifteen degrees. For example, when it is noon in Cape May, New Jersey, it will be exactly 9:00 A.M. in Santa Barbara, California, and Heppner, Oregon and Ephrata, Washington, since all of these cities are 45° west of Cape May. Of course in our day-to-day world, we don't worry about being so exact. For convenience we define time zones and assign all cities within each zone the same time, even though they are not all on the same longitude line.

A British navigator of the nineteenth-century, whose clock was set to London time, had no problem determining his longitude. Regardless of what happened at sea, he needed to only note the position of the sun and its relationship to the time on his clock. If, for example, the sun was at high noon in his present location and his clock read 3:30 P.M., he was three and one-half hours or 52.5° (15°/hour × 3.5 hours) west of London. If his latitude hadn't changed since he set course, this would place him near the coast of Newfoundland.

* * *

Exploration and colonization instilled greater life into trade, commerce, and technology for Southern and Western Europe during the sixteenth, seventeenth, and eighteenth centuries. Renewed monetary systems that had begun in thirteenth-century Florence and Genoa spread to all of Europe as commerce escalated during the Renaissance and Age of Discovery. Banking evolved and coinage circulated freely once again in Europe. The commercial realities of the old East and new West brought Europe fully out of the Dark Ages. Mathematical literacy took on importance again as the societies of Western Europe matured into economic world centers. Dennis Richardson gives us a hint of the financial atmosphere of Europe in the early seventeenth century.

> In 1609, the bank of Amsterdam was organized to give the community relief from worn and defaced coins. One function of the bank was to accept for safe keeping in perpetuity gold and silver coins and to make credit transfers from one account to another on written orders.
>
> Coins were deposited with the bank at a discount of 5 percent, and the depositor was charged a service fee of ten florins to cover the costs of opening his account.[32]

Obviously, those who led a comfortable life were fluent in the language of investing, which was written in mathematics.

* * *

Literacy was advanced in the West with the invention of the printing press. This wonderful machine made books plentiful and inexpensive, in contrast to the mediæval period, when books were rare and expensive. It was the first time that Greek thought could be read, debated, and analyzed by an evergrowing literate population.

By the mid-sixteenth century, the Polish astronomer, Nicolaus Copernicus (1473-1543) had refuted Aristotle (384 B.C.-322 B.C.) and Ptolemy (85 A.D.-165 A.D.) by advancing the heliocentric (sun centered) view as opposed to their geocentric (earth centered) view of the solar system. Mathematics was also advancing at a respectable

pace. John Napier (1550-1617), a Scot, is credited with the invention of logarithms. You may remember studying logarithms if you had Algebra II in high school or college mathematics. However, if you're like most of the well-educated people I have spoken to, logarithms mean absolutely nothing to you!

Logarithms are a convenient mathematical device that eases computations with large numbers by transforming multiplication and division into addition and subtraction. They can also be used to "linearize" nonlinear relationships. Few people had use for logarithms in the seventeenth century. But those who did, such as Johann Kepler (1571-1630) and Isaac Newton (1642-1727), helped to speed along the scientific and mathematical revolutions that have altered how we live and what we need to know.

The Scientific Revolution would not have been possible without the scientific method. Whether we realize it or not, the scientific method has become a part of the way we think. When we hear something remarkable, we usually ask that the statement be supported with facts. Facts, however, are based on observations, and depending on the situation, experimentation. Today, even to the darkest mind, experimentation has value. On the other hand, the typical Medieval person would not have valued the process of experimentation as you or I do. We have been indoctrinated to verify; the process used to do this is the scientific method—formulating a problem and using experimental techniques to come to a conclusion. It is the method that Galileo Galilei (1564-1642) established as the cornerstone of his research and which every other rational thinker has used since.

Even when the results seem incredible, we believe in the truth of scientific results because of the scientific method. Consider the theories of Einstein. Unless you have taken a physics course, you probably do not know what his Special or General Theories of Relativity state. Yet most people see Einstein as a twentieth century icon for science. Why? Because the scientific community tells us that his theories yield correct results when applied to real-world problems.

One of the reasons Einstein's work has such a mystique, is that it violates our common sense view of the world. Einstein tells us that space and time are not absolute quantities. According to Einstein, every time we travel in our car, we begin to age more slowly, increase our mass, and shrink in the direction of motion we are moving in—relative to someone sitting by the side of the road. But the only way to notice these effects is to move unimaginably fast. We would have to accelerate to an appreciable percentage of the speed of light, 186,000 miles per second, in order to detect these changes without sophisticated machinery.

Scientists have conducted experiments with particles which have so little mass they can be accelerated to 99 percent of the speed of light. These experiments verify that the particle's mass has increased and its "life span" is longer. Some particles "live" for only a fraction of a second before decaying into other particles. Measuring the time for a particle to decay when it is at rest relative to us, as opposed to when it is moving at a velocity close to light, shows a time difference in the decay rates.

Another experiment, also showing the effects of "time dilation,"(the slowing of time) uses an atomic clock. Atomic clocks are incredibly accurate devices that use the frequency of atomic vibrations to measure time. They lose no more than one second in a million years. This clock is placed aboard a high speed jet and an identical clock that is synchronous with the clock on the jet remains on the ground. When the jet returns the two clocks are out of sync, the clock on the jet reading behind the clock on the ground, by just the amount Einstein's theory predicts. Hundreds of experiments like these have been performed since Einstein published his Special Theory of Relativity in 1905, and each one has verified the theory. As a matter of fact, scientists could not explain their results or carry on other experiments if they did not use Einstein's theories.

* * *

On the coattails of the Scientific Revolution was the Industrial

Revolution. With the advent of mechanization, Western Europe and the United States shifted from basically an agrarian to urban life style. A new social class developed—the wage earner; with him came the time clock, personal income taxes, and the birth of the modern consumer. Numeracy became essential, not a luxury reserved for the well-to-do. Seventh-grade mathematics was finally in everyone's hands.

In the early twentieth century it was algebra, geometry, and trigonometry that separated those who were headed for white collar jobs from those who needed only "business math." Such lines of separation are always fuzzy and many self-made millionaires have been clever enough to prosper without algebra. It will be rare, however, to find millionaires in the twenty-first century not exposed to calculus. Mathematics, in many respects, is becoming the new language for humanity. Social, political, and economic forces are compelling us to become more numerate.

In prior centuries the focus of mathematics was outward; its purpose was to explain the universe. Today we are turning it upon ourselves, gathering data on mass human behavior, trying to understanding how we function, and what we will require in the future.

It has only been within the last two centuries that participatory government has embraced large populations. The growing demand for an educated and numerate society is imperative in an age where people must decide which direction the world will take on the issues of environment, universal health care, species extinction, free trade agreements, foreign aid, population control, nuclear proliferation, and devastating diseases such as AIDS. These problems have a quantitative component that cannot be ignored.

Only by understanding the consequences of our actions, can we change our world for the better. Mathematics can help us see where we are going and what is possible. The responsibility of then acting on those possibilities rests with our political and social will.